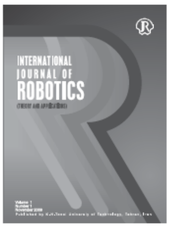

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Study of Lateral Vibrations of a Prismatic Actuator and Its Effects on the Accuracy of the 6UPS Stewart Robot

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ABSTRACT

Many robotic mechanisms rely on rotating linear (prismatic) actuators, whose performance and accuracy can be significantly affected by lateral vibration. In this study, the vibrational behavior of a flexible rotating prismatic actuator and its impact on the end-effector accuracy of a 6UPS Stewart robot are investigated. First, an analytical model is developed using the assumed modes method to evaluate the lateral vibration of a single actuator, and its results are compared with a finite element (FEM) analysis in ABAQUS. The close agreement between the two approaches confirms the validity of the numerical model. Next, the entire 6UPS Stewart robot is simulated with elastic actuators, and the resulting trajectory of the end effector (or center of mass of the payload) is compared against the corresponding rigid-robot motion. The analyses show that actuator flexibility and vibration can introduce noticeable deviations; reaching up to 6% in the end-effector's position; highlighting the need to account for vibrational effects in the design and control of high-precision parallel robots.

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1. Introduction

Due to their geometric structure, parallel robots—such as the 6UPS Stewart robot—exhibit higher stiffness and accuracy compared to many other robotic systems, making them particularly suitable for precision applications. Most parallel robots incorporate prismatic actuators in their design. The demand for fast and agile robots has led to the use of lightweight prismatic actuators. However, making these actuators lightweight also reduces their rigidity and increases vibrations, which can negatively affect the robot's accuracy and overall performance. These vibrations arise from the elastodynamic properties of the mechanisms, highlighting the importance of investigating the oscillatory behavior of prismatic actuators in the design and control of parallel robots[1]. Generally, elastodynamic models of robots are developed using lumped modeling approaches or by incorporating distributed flexibilities [2].

The lumped modeling approach is the simplest method for representing these actuators; however, it involves significant simplifications and thus limited accuracy, making it less suitable for high-precision systems. For example, Megahed and Hamza [3] applied a modified version of finite segment multibody dynamics to a surface elastic actuator whose tip was connected to a revolute joint. Their formulation used a solid mass matrix to improve accuracy compared to traditional multi-section lumped methods. Kim [4] proposed an equivalent lumped-element system to be used in the controller of an elastic actuator, converting the actuator into a coupled system via impedance modeling. Their study explored the impact of both series and parallel configurations of springs and dampers.

Most analytical models for the dynamics of elastic parallel robots still rely on lumped parameter representations. For instance, Lee and Geng [5] derived 12 Lagrange equations for a flexible Stewart platform manipulator using a tensor-based approach. Although the mass of the actuators was distributed, the stiffness and damping were considered as lumped values. Similarly, Mokhergi et al. [6] analyzed the dynamics and vibrations of a Stewart platform by modeling actuator stiffness, frictional forces, and torques as lumped parameters, focusing primarily on axial stiffness. Afzalifar et al. [7] developed a fully parametric dynamic model for damped vibrations of a Stewart robot in a symmetric configuration, also accounting for actuator inertia; however, their approach inherited the same fundamental limitations as Mokhergi et al.

In contrast, modeling with distributed flexibility offers higher accuracy in representing the elastic behavior of actuators. Yet, the resulting equations become very complex and are challenging to apply when multiple actuators must be considered in a parallel robot. Consequently, this approach is more commonly used for simpler mechanisms or single actuators. In this context, Pan et al. [8] presented a dynamic model of elastic prismatic actuators, employing conditional kinematics for prismatic and revolute joints. They demonstrated that small deformations in the actuator due to flexibility could be successfully captured in a numerical example. Tadikonda and Baruh [9] developed a complete dynamic model for a beam with a tip mass and reciprocating motion in a rigid hub, showing that neglecting flexibility can significantly compromise positional accuracy. Similarly, Khulief and Bedoor [10] introduced a dynamic model for a flexible beam with prismatic and revolute joints using a Lagrangian approach combined with the assumed modes method. Yoksel and Gurgoze [11] analyzed bending stresses in a robotic arm during linear motion based on Euler-Bernoulli beam assumptions, taking gravity effects, rotary inertia, and axial forces into account. Khadem and Pirmohammadi [12] proposed a mathematical model for robotic arms with prismatic and revolute joints operating in three-dimensional space, using perturbation methods for deriving vibrational equations. Kalyoncu [13] presented a dynamic model that could follow a piecewise straight path under external forces and torques. Wang et al. [14] derived lateral vibration equations for an Euler-Bernoulli beam to estimate the coupling between translational motion and lateral vibration. Nagarjan and Turcic [15] provided systematic equations for mechanisms with both rigid and flexible links. Chang and Gannon [16] produced an enhanced equivalent rigid link model using natural mode shapes for flexible actuators, employing both Lagrange and finite element methods. Briot and Khalil [17] developed a symbolic and recursive method for the elastodynamic modeling of flexible parallel robots by combining a generalized Newton–Euler model with the principle of virtual powers; however, their approach is computationally intensive for a 6UPS Stewart platform. Noorian and Akbarzade [18] have presented a model for a prismatic actuator used in a parallel robot. In their research, the authors derive the direct dynamic equations for a rotating prismatic actuator featuring three degrees of freedom in the axial direction and a ball screw drive system, employing the Lagrange method. They take into account not only the flexibility of the moving piston but also model the ball screw with variable stiffness. A significant aspect of the study is the changing stiffness of the ball screw. As the nut travels along the shaft, both the active length and the shaft's stiffness vary, closely mimicking real-world conditions.

Alongside analytical methods, there has been substantial research utilizing finite element methods (FEM) to study elastic actuators. Moulin and Bayo [19, 20] used FEM to explore end-point trajectory tracking in flexible actuators. Tokhi et al. [21–24] proposed a dynamic model of an elastic single-link actuator using FEM, validating their model by comparing the numerically obtained modal frequencies with experimental results. Li and Sankar [25] investigated a three-dimensional single-link elastic actuator using FEM.

In this study, the behavior of a flexible rotating prismatic actuator with a tip mass—representative of those in a 6UPS Stewart robot—is first examined via both analytical and numerical methods. The analytical approach uses the Rayleigh-Ritz method to capture vibrations in the distributed elastic beam at the core of the actuator. The resulting equations are computed parametrically, and a reference geometry and initial conditions are used to determine the vibrational response. For simplicity, the analytical model is treated in two dimensions. A three-dimensional model of this actuator is then simulated in ABAQUS, ensuring the motion and lateral vibrations occur in a single plane so that it can be directly compared with the 2D analytical results. Having validated the finite element method, a full 3D FEM model of the 6UPS Stewart robot with elastic prismatic actuators is subsequently created and analyzed.

The main contributions of this study are:

- A comprehensive analytical and numerical investigation of the vibrational behavior of elastic prismatic actuators within a 6UPS Stewart robot.
- Validation of finite element models through comparison with analytical results derived from the assumed modes method, ensuring the reliability and accuracy of the simulations.
- Quantification of the impact of actuator flexibility on the end-effector's positional accuracy, demonstrating deviations of up to 6%.
- Establishment of a robust FEM-based framework for the elastodynamic analysis of complex parallel robotic systems, facilitating future design and control optimizations.

Table 1 Classification of literature introduced in introduction

Category	Author(s)	Main Contribution
Lumped Modeling Approaches	Megahed & Hamza [3]	Developed a multi-body dynamics model for elastic actuators with improved accuracy.
	Kim [4]	Proposed a lumped-element system for actuator control using impedance modeling.
	Lee & Geng [5]	Derived dynamic equations for flexible Stewart platforms with lumped stiffness.
Distributed Flexibility Approaches	Pan et al. [8]	Modeled elastic actuators capturing small deformations, validated numerically.
	Tadikonda & Baruh [9]	Created a dynamic model for flexible beams with tip masses, showing errors if neglected flexibility.
	Khulief & Bedoor [10]	Developed vibrational models for flexible beams using assumed modes method.
	Noorian & Akbarzadeh [18]	Developed of a specialized model for a prismatic actuator tailored for use in parallel robots, thereby enhancing their performance and functionality.
Finite Element Method Applications	Moulin & Bayo [19, 20]	Used FEM to study trajectory tracking in flexible actuators.
	Tokhi et al. [21–24]	Modeled elastic actuators with FEM, validating against experimental frequencies.
	Li & Sankar [25]	Investigated 3D elastic actuators using FEM.
Inverse Dynamics for Rigid Robots	Dasgupta & Mruthyunjaya [27]	Developed inverse dynamic equations for rigid Stewart robots.

2. Vibrational model of an elastic prismatic actuator

Figure 1 presents a schematic of a two-degree-of-freedom elastic actuator with a point mass attached at its tip. This actuator is representative of each actuator used in the Stewart robot. A torque TTT is applied to the rigid hub. The following assumptions are made:

1) The XY coordinate system is the inertial frame of reference, while the xy coordinate system is attached to the hub and rotates about a fixed-point O.
2) The rigid hub has a rotational inertia moment denoted by J_h, and the beam is free to move within the hub without friction.

3) The beam is sufficiently long and slender to justify the use of Euler–Bernoulli beam theory and is assumed to be inextensible.

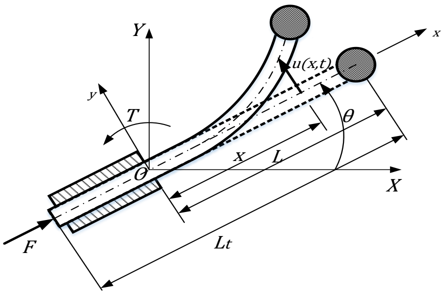

Figure 1. The schematic view of the elastic actuator

2.1. Analytical approach in vibrations of the elastic prismatic actuator

In the analytical approach, the displacement and vibration of the flexible arm tip, which moves within a rigid hub, were calculated using the assumed modes method, following Khulief and Bedoor [10]. The primary equation governing the flexible motion in the assumed modes framework is given by Equation (1).

$$u(x,L,t)=\sum_{i=l}^{k}Y_i(x,L)\,u_i(t) \tag{1}$$

Where, $Y_i(x,L)$ is the assumed mode function, u_i is the modal coordinate, L is the length of the beam and k shows the number of modes. In the Eq. 1 length of the beam is a function of time. The special functions of the cantilever beam were used as the assumed mode shapes. This functions were extracted in Eq. 2, after normalization based on mass per unit length of the beam (ρ).

$$\begin{gathered}Y_i(x,L)=\frac{1}{\sqrt{\rho L}}\left[\sin\epsilon_i\frac{x}{L}-\sinh\epsilon_i\frac{x}{L}\right.\\ \left.-\alpha_i\left(\cos\epsilon_i\frac{x}{L}-\cosh\epsilon_i\frac{x}{L}\right)\right]\\ \alpha_i=\frac{\sin\epsilon_i+\sinh\epsilon_i}{\cos\epsilon_i+\cosh\epsilon_i}\\ \epsilon_i=\beta_i L\\ \beta_i^4=\frac{\omega_i^4\rho}{EI}\end{gathered} \tag{2}$$

Where, ω_i is the ith natural frequency, E is the elastic module of the beam, and I is the mass inertia moment of the beam. ϵ_i is defined as transcendental Eq. 3.

$$1+\cos\epsilon_i\cosh\epsilon_i=0 \tag{3}$$

Before using the Lagrange equation of motion, the kinematic and the potential energy of the system must be calculated. Eq. 4 shows the equation of kinematic energy of an elastic beam.

$$U_b=\frac{1}{2}\int\dot{R}_p^t\dot{R}_p\,dm \tag{4}$$

Where, $\dot{R}_p$ is the velocity of physical point p inside the beam. The kinematic energy of the point mass in the tip of the beam is as Eq. 5.

$$\begin{gathered}U_e=\frac{1}{2}m_e\dot{R}_e^T\dot{R}_e=\frac{1}{2}m_e\big[\big(L^2+u^2(L)\big)\dot{\theta}^2+\dot{L}^2\\ +u^2(L)+m_eL\dot{\theta}u(L)\big]\end{gathered} \tag{5}$$

Where, m_e is the mass at the tip of the beam and $\dot{R}_e$ is its velocity. The total kinematic energy of system is the summation of Eq. 4 and Eq. 5.

The potential energy of system consists of three parts: i) elastic strain energy which is conserved at the flexible beam; ii) gravity potential energy; iii) strain energy due to centrifugal force in rotary motion. Flexible strain energy is as Eq. 6.

$$V_s=\frac{1}{2}\int EI(x)\left[\frac{\partial^2y}{\partial x^2}\right]^2dx \tag{6}$$

The gravitational potential energy of beam and tip mass is as Eq. 7.

$$\begin{aligned}V_g&=\int_{L-L_i}^{L}\rho gxsin\theta\,dx+m_egLsin\theta\\ &=\rho gxsin\theta\left(LL_t-\left(\frac{L_t^2}{2}\right)\right)+m_egLsin\theta\end{aligned} \tag{7}$$

The strain energy due to centrifugal force in the rotation of beam is as Eq. 8. Also, the axial forces F_p and F_e can be expressed in Eq. 9 and Eq.10, respectively. The total potential energy of the system is the summation of Eq. 6, 7 and 8.

$$V_r=\frac{1}{2}\int_0^L F_p\left[\frac{\partial u}{\partial x}\right]^2dx+F_eL\left[\frac{\partial u}{\partial x}\bigg|_L\right]^2 \tag{8}$$

$$F_p=\int\rho\dot{\theta}^2xdx=\frac{\rho\dot{\theta}^2}{2}(L^2-x^2) \tag{9}$$

$$F_e=m_e\dot{\theta}^2L \tag{10}$$

Finally, by using the variable state of equations of Lagrange motion, $k+2$ coupled nonlinear ordinary differential equation were obtained [10]. This dynamic model has $k+2$ degrees of freedom which include $L,\theta,u_r\ \ (r=1,\dots,k)$. With considering three modes (k=3), the lateral vibrations of the beam were calculated with the specifications mentioned in the Table 2.

Table 2 Information of geometrical model of the beam in elastic actuator

Property	Quantity
Total length of the beam	3.66 m
Cross section of the beam	15.24×0.95 cm^2
Mass per unit length of the beam	4.02 kg/m
Bending rigidity of the beam	756.65 Nm^2

2.2. Finite element investigation in vibrations of prismatic actuator

A three-dimensional prismatic actuator, with the properties listed in Table 1, was modeled in the finite element software ABAQUS. Figure 2 illustrates the corresponding CAD model. The main beam was modeled as a solid, deformable component, while the hub was defined as a rigid shell. The contact between the hub and the beam was assumed to be frictionless.

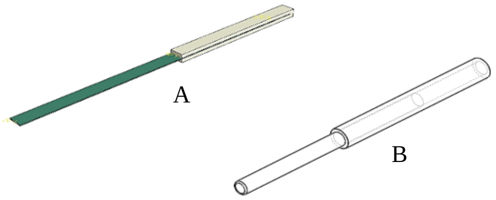

Figure 2. The simplified prismatic actuator analyzed analytically and with FEM (A), the actuator used in Stewart 6UPS robot

At the start of the simulation, the beam's tip was displaced by 5 mm to introduce an initial disturbance for vibration. Figure 3 compares the vibratory response of the beam using both coarse and fine meshes. Mesh refinement was carried out from 5 cm down to 5 mm. At a mesh size of 5 mm for cubic elements, the simulation results reached convergence.

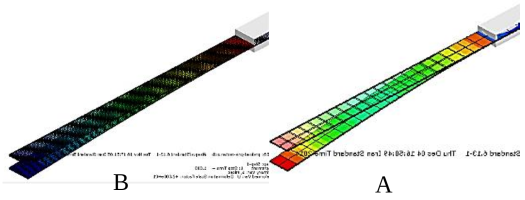

Figure 3. The vibrations of elastic actuator in ABAQUS, coarse mesh (A), fine mesh (B)

Figure 4 compares the lateral vibration at the beam's tip using both analytical and numerical approaches. These results were generated with a linear velocity of 0.1 m/s and a rotational velocity of 0.785 rad/s. The close agreement between the two methods demonstrates that the analytical approach aligns well with the numerical findings.

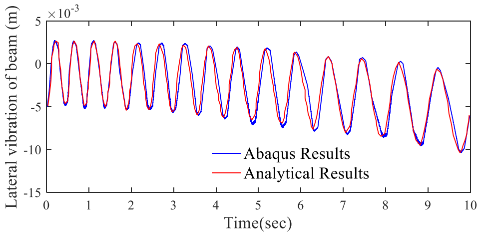

Figure 4. The comparison of numerical and analytical results for vibrations of the elastic actuator

3. Vibrations of the 6UPS Stewart robot with flexible prismatic actuators

Due to the complexity of dynamic analytical modeling for flexible continuous systems, a numerical approach such as Finite Element Method (FEM) simulation is often an effective alternative. In this study, a reference Stewart robot was analyzed numerically to evaluate how actuator flexibility influences the positioning accuracy of the robot's end effector. Figure 5 presents the CAD model of this robot, where each actuator is connected to the base platform via a universal joint and to the moving platform via a spherical joint. The physical properties of this model are provided in Table 3. Because the ratio of piston diameter to length is less than ten percent, the assumptions of Euler–Bernoulli beam theory remain valid. [26].

To ensure accurate modeling of the boundary conditions and joints in the Stewart robot, its motion was analyzed with rigid actuators using both analytical and numerical methods. For the rigid configuration, the inverse dynamic equations prescribed by Dasgupta and Mruthyunjaya [27] were employed to validate the simulation results. In the analytical approach, a specified Cartesian-space trajectory served as the input, while the corresponding axial displacement and force profiles for each prismatic actuator were determined in Joint space. The chosen trajectory, termed the "Roll" trajectory, involves rotation of the end effector about the X-axis, as illustrated in Figure 6. In ABAQUS, these displacements—calculated from the rigid robot's inverse dynamic equations—were applied to each of the robot's actuators. Figure 7 shows the robot's configuration at the end of the Roll trajectory, where the pistons achieve a maximum travel of 260 mm.

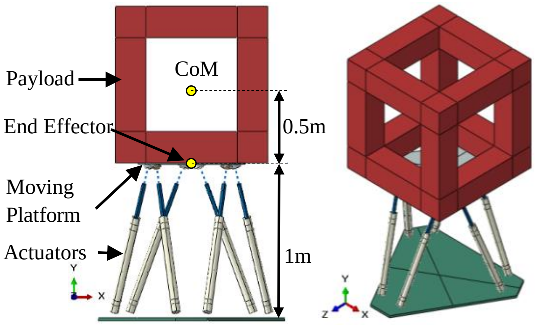

Figure 5. The CAD model of the reference 6UPS Stewart robot

Table 3. Physical properties of the reference 6UPS Stewart robot

Parameter	Quantity
Payload mass	100kg
Payload moment inertias I_{xx}, I_{yy}, I_{zz}	17.88 kgm^2
Diameter of the main circle of base platform	0.5m
Diameter of the main circle of moving platform	0.3m
Piston Diameter	27mm
Elastic length of piston	600mm
Elastic modulus of piston	200GPa
Density of piston	7800Kg/m3
Gravitational acceleration	9.81m/s2

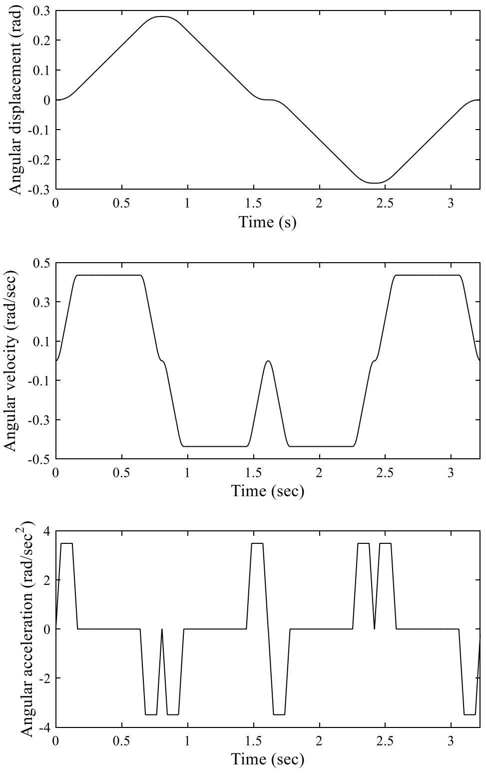

Figure 6. The properties of roll trajectory (rotation about X axis) for end effector of reference robot

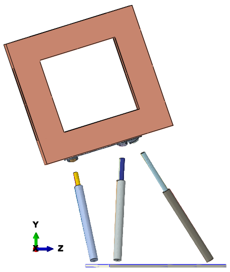

Figure 7. Reference Stewart robot at the end of roll trajectory

3.1. Rigid dynamic analysis of 6UPS Stewart robot

There are several methods for analyzing the rigid-body dynamics of robotic systems. Among these, the Newton–Euler and Euler–Lagrange methods are commonly applied to derive and examine the dynamic equations for parallel robots. Unlike the Euler–Lagrange approach, which relies on sequential differentiation, the Newton–Euler method can directly yield the dynamic equations once the linear and rotational velocities and accelerations of the robot are determined [2]. Therefore, the Newton–Euler method proves more efficient for computing the inverse dynamics of parallel robots. In this study, using the Newton–Euler formulation proposed in [27], the forces and torques of the robot's prismatic actuators were derived for the reference Roll trajectory, treating the 6UPS Stewart robot as a rigid system.

In parallel, a finite element analysis was performed on the same rigid reference robot, and the variations in axial force for each actuator were obtained. Figure 8 compares the actuator forces from both the analytical and numerical approaches. The strong agreement between the two sets of results confirms the accuracy of the FEM modeling, enabling a reliable transition to finite element simulations of the flexible robot.

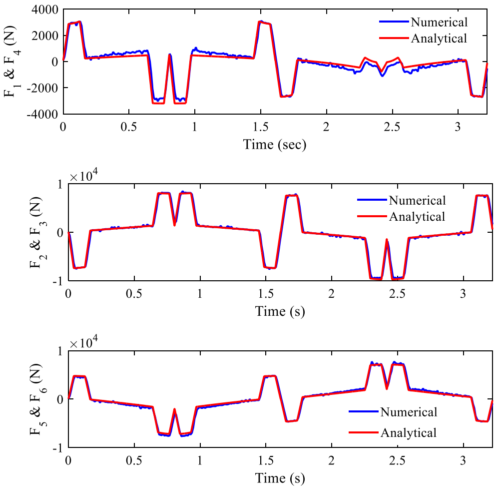

Figure 8. The comparison of the actuator forces between analytical and numerical approaches

3.2. Elastodynamic analysis of 6UPS Stewart robot

The finite element analysis of the reference robot, now equipped with elastic actuators, was performed in ABAQUS. The end effector followed the previously introduced Roll trajectory, with the pistons' elastic properties specified in Table 2. From a user perspective, the critical output was the trajectory of the payload's center of mass (CoM), where positional accuracy is paramount. Under rigid conditions, the desired CoM path aligns with Equations (11) and (12) in the global Y and Z directions, respectively.

$$Y = dcos(\theta) \qquad (11)$$

$$Z = dsin(\theta) \qquad (12)$$

Here, d represents the direct distance between the CoM and the end effector, while θ is the roll position, which changes over time as shown in Figure 6. Figure 9 compares the CoM trajectory for both the rigid and flexible versions of the robot.

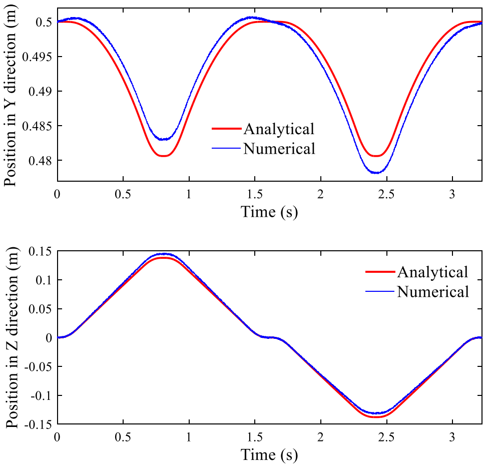

Figure 9. Comparison of CoM path in Y and Z directions

Figure 10 depicts the robot equipped with elastic actuators at the maximum roll angle, illustrating the excited mode of one of the pistons.

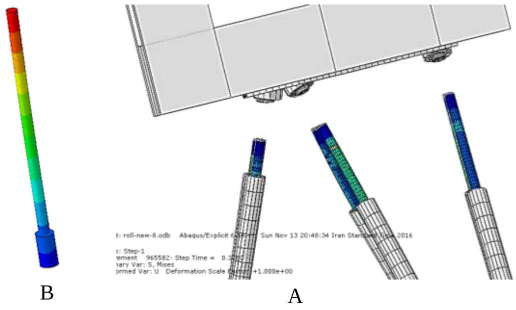

Figure 10. Robot with elastic actuators in ABAQUS environment (A), stimulated mode of pistons (B)

4. Conclusions

In this study, we conducted a comprehensive investigation into the vibrational behavior of elastic prismatic actuators within a 6UPS Stewart robot, employing both analytical and numerical methodologies. The analytical model utilized the assumed modes method to capture the dynamic response of a single flexible actuator, while finite element simulations in ABAQUS provided a robust numerical framework for analyzing the entire robotic system.

This study successfully demonstrates the critical role of actuator flexibility in the dynamic performance of parallel robots and validates the use of finite element simulations as a powerful tool for elastodynamic analysis. By quantifying the impact of elasticity on end-effector accuracy, the research provides a foundation for future advancements in the design and control of high-precision robotic systems. Emphasizing the integration of accurate modeling techniques with innovative design strategies will be essential for developing next-generation parallel robots capable of performing complex tasks with enhanced reliability and precision.

4.1. Key Findings

1. **Validation of Analytical and Numerical Models:**
   The comparison between the analytical results and finite element simulations demonstrated a strong agreement, thereby validating the accuracy and reliability of the FEM approach for modeling elastic actuators. This confirmation is crucial, as it establishes confidence in using FEM for more complex and comprehensive analyses of parallel robotic systems.
2. **Impact of Actuator Flexibility on Accuracy:**
   The finite element simulations revealed that actuator flexibility induces significant positional deviations in the robot's end effector during motion. Specifically, during the Roll trajectory, we observed deviations of approximately **5mm** along the Y-axis and **10mm** along the Z-axis. These findings underscore that even though parallel robots like the 6UPS Stewart are inherently stiff, the elastic properties of their actuators can lead to considerable inaccuracies in precision tasks.
3. **Advantages of FEM in Elastodynamic Analysis:**
   The study highlights the effectiveness of finite element simulations in handling the elastodynamic complexities of parallel robots. Unlike traditional lumped modeling approaches, FEM can accurately capture the distributed flexibilities and dynamic

interactions within the robotic mechanisms, making it a superior tool for detailed analysis and optimization.

4.2. Implications of the Study

The results of this research have significant implications for the design and control of high-precision parallel robots. Understanding the extent to which actuator flexibility affects end-effector accuracy allows engineers to make informed decisions regarding actuator selection, material properties, and structural design to mitigate unwanted vibrations. Additionally, the validated FEM framework can be utilized to explore various design modifications and control strategies aimed at enhancing the overall performance and reliability of parallel robotic systems.

4.3. Limitations

While the study provides valuable insights, it is not without limitations. The current analysis focused on a specific type of parallel robot (6UPS Stewart) and a particular actuator configuration. Future studies should explore a wider range of robotic architectures and actuator designs to generalize the findings. Moreover, the simulations assumed ideal conditions without accounting for factors such as joint friction, material imperfections, and external disturbances, which can further influence vibrational behavior and accuracy.

4.4. Future Research Directions

Future research could focus on developing active vibration control methods, such as real-time feedback systems or adaptive damping strategies, to effectively mitigate the effects of actuator flexibility. Implementing these control mechanisms would enhance the positional accuracy and overall performance of parallel robots by actively counteracting unwanted vibrations. Additionally, exploring optimized actuator designs by investigating new materials and structural configurations can lead to actuators with higher stiffness-to-weight ratios. This would reduce vibrational impacts without compromising the robot's agility and responsiveness, thereby improving both precision and efficiency in various applications.

Another promising avenue is the integration of multiphysics simulations that incorporate thermal, fluidic, or electromagnetic effects alongside structural dynamics. This holistic approach would provide a more comprehensive understanding of actuator performance under diverse operational conditions. Furthermore, conducting experimental validations through the development of physical prototypes will bridge the gap between theoretical models and practical implementations, ensuring that simulation results accurately reflect real-world behavior. Lastly, advancing control algorithms to account for and compensate elastic deformations will further enhance the precision and reliability of parallel robots, enabling them to perform complex tasks with greater accuracy and consistency.

Nomenclature

Abbreviations

6UPS: A parallel robot configuration with six Universal–Prismatic–Spherical joints

CAD: Computer-Aided Design

CoM: Center of Mass

FEM: Finite Element Method

Latin Letters and Functions

x: Spatial coordinate along the beam’s length

L: Length of the beam

k: Number of modes in the assumed-modes method

$u(x,L,t)$: Transverse displacement of the beam in the assumed-modes method

$Y_i(x,L)$: i-th assumed mode shape function of the beam

$u_i(t)$: Modal coordinate for the i-th mode

m_e: Tip mass at the beam’s free end (end effector/payload)

R_p, R_e: Position vectors for a point p on the beam and the tip mass, respectively

$\dot{R}_p, \dot{R}_e$: Velocity vectors for point p on the beam and the tip mass, respectively

t: Time

E: Young’s (elastic) modulus of the beam

I: Second moment of area (area moment of inertia) of the beam cross-section

U_b: Kinetic energy of the beam

U_e: Kinetic energy of the tip mass (end effector)

V_s: Strain energy in the beam due to bending

V_g: Potential energy due to gravity

V_r: Strain energy due to centrifugal force in rotary motion

F_p: Distributed axial (centrifugal) force along the beam

F_e: Axial (centrifugal) force at the tip mass

g: Gravitational acceleration

T: Torque (moment) applied to the rigid hub

J_h: Rotational inertia of the hub

Greek Letters

ρ: Mass per unit length (linear density) of the beam

α_i: A function appearing in the cantilever beam mode shapes

ε_i: Dimensionless parameter in the mode-shape equations

β_i: Parameter related to the i-th eigenvalue or mode shape;

ω_i: i-th natural frequency of the beam (may also denote angular velocity, depending on context)

θ: Roll angle or rotational coordinate

$\dot{\theta}$: Angular velocity (time derivative of θ)

References

[1] S. K. Dwivedy and P. Eberhard, “Dynamic analysis of flexible manipulators, a literature review,” *Mech. Mach. Theory*, vol. 41, no. 7, pp. 749–777, 2006.

[2] S. Briot and W. Khalil, *Dynamics of Parallel Robots*, vol. 35. Cham: Springer International Publishing, 2015.

[3] S. M. Megahed and K. T. Hamza, “Modeling and simulation of planar flexible link manipulators with rigid tip connections to revolute joints,” *Robotica*, vol. 22, no. May 2004, pp. 285–300, May 2004.

[4] S. M. Kim, “Lumped element modeling of a flexible manipulator system,” *IEEE/ASME Trans. Mechatronics*, vol. 20, no. 2, pp. 967–974, 2015.

[5] J. D. Lee and Z. Geng, “A dynamic Model of A Flexible Stewart Platform,” *Comput. Struct.*, vol. 48, no. 3, pp. 367–374, 1993.

[6] P. Mukherjee, B. Dasgupta, and A. K. Mallik, “Dynamic stability index and vibration analysis of a flexible Stewart platform,” *J. Sound Vib.*, vol. 307, no. 3–5, pp. 495–512, 2007.

[7] B. Afzali-far, A. Andersson, K. Nilsson, and P. Lidström, “Influence of strut inertia on the vibrations in initially symmetric Gough – Stewart Platforms — an analytical study,” *J. Sound Vib.*, vol. 352, pp. 142–157, 2015.

[8] Y.-C. Pan, R. a. Scott, and a. G. Ulsoy, “Dynamic Modeling and Simulation of Flexible Robots With Prismatic Joints,” *J. Mech. Des.*, vol. 112, no. September 1990, p. 307, 1990.

[9] S. S. K. Tadikonda and H. Baruh, “Dynamics and Control of a Translating Flexible Beam With a Prismatic Joint,” *J. Dyn. Syst. Meas. Control*, vol. 114, no. 3, p. 422, 1992.

[10] Y. Khulief, B. Bedoor, “Vibrational motion of an elastic beam with prismatic and revolute joints,” *J. Sound Vib.*, vol. 190, no. 2, pp. 195–206, 1996.

[11] Ş. Yüksel and M. Gürgöze, “On the flexural vibrations of elastic manipulators with prismatic joints,” *Comput. Struct.*, vol. 62, no. 5, pp. 897–908, 1997.

[12] S. E. Khadem and A. A. Pirmohammadi, “Analytical development of dynamic equations of motion for a three-dimensional flexible link manipulator with revolute and prismatic joints,” *IEEE Trans. Syst. Man, Cybern. Part B Cybern.*, vol. 33, no. 2, pp. 237–249, 2003.

[13] M. Kalyoncu, “Mathematical modeling and dynamic response of a multi-straight-line path tracing flexible robot manipulator with rotating-prismatic joint,” *Appl. Math. Model.*, vol. 32, no. 6, pp. 1087–1098, 2008.

[14] L. H. Wang, Z. D. Hu, Z. Zhong, and J. W. Ju, “Hamiltonian dynamic analysis of an axially translating beam featuring time-variant velocity,” *Acta Mech.*, vol. 206, no. 3–4, pp. 149–161, 2009.

[15] S. Nagarajan and D. A. Turcic, “Lagrangian Formulation of the Equations of Motion for Elastic Mechanisms With Mutual Dependence Between Rigid Body and Elastic Motions: Part I—Element Level Equations,” *J. Dyn. Syst. Meas. Control*, vol. 112, no. 2, pp. 203–214, Jun. 1990.

[16] L.-W. Chang and K. P. Gannon, “A Dynamic Model on a Single-Link Flexible Manipulator,” *J. Vib. Acoust.*, vol. 112, no. 1, p. 138, Jan. 1990.

[17] S. Briot and W. Khalil, “Recursive and symbolic calculation of the elastodynamic model of flexible parallel robots,” *Int. J. Rob. Res.*, vol. 33, no. 3, pp. 469–483, Mar. 2014.

[18] A. Noorian, A. Akbarzadeh , Analytical dynamic of a rotating prismatic actuator with variable stiffness and validation with finite element method, Modares Mechanical Engineering, Vol. 17, No. 3, pp. 7 9 -86, 2017 (in Persian)

[19] H. Moulin and E. Bayo, “On the Accuracy of End-Point Trajectory Tracking for Flexible Arms by Noncausal Inverse Dynamic Solutions,” *J. Dyn. Syst. Meas. Control*, vol. 113, no. 2, pp. 320–324, Jun. 1991.

[20] H. C. Moulin and E. Bayo, “Accuracy of discrete models for the inverse dynamics of flexible arms, feasible trajectories,” in *Proc. of the 29th IEEE Conf. on Decision and Control*, 1990, vol. 2, no. December, pp. 531–532.

[21] M. O. Tokhi, Z. Mohamed, J. M. Martins, M.

A. Botto, and J. Sa da Costa, “Approaches for dynamic modeling of flexible manipulator systems,” in *IEE Proceedings - Control Theory and Applications*, 2003, vol. 150, no. 4, pp. 401–411.

[22] J. Martins, M. Ayala Botto, and J. S. Da Costa, “Modeling of flexible beams for robotic manipulators,” *Multibody Syst. Dyn.*, vol. 7, no. 1, pp. 79–100, 2002.

[23] M. O. Tokhi, “A Simulink environment for simulation and control of flexible manipulator systems,” in *UKACC International Conference on Control. Control ’96*, 1996, vol. 1996, pp. 210–215.

[24] M. Khairudin, Z. Mohamed, A. R. Husain, and R. Mamat, “Dynamic characterization of a two-link flexible manipulator: theory and experiments,” in *Advances in Robotics Research*, 2014, vol. 1, no. 1, pp. 61–79.

[25] C. J. Li and T. S. Sankar, “Systematic Methods for Efficient Modeling and Dynamics Computation of Flexible Robot Manipulators,” *IEEE Trans. Syst. Man Cybern.*, vol. 23, no. 1, pp. 77–95, 1993.

[26] O. Bauchau and J. Craig, “Euler-Bernoulli beam theory,” in *Structural Analysis*, Springer Netherlands, 2009, pp. 173–221.

[27] B. Dasgupta and T. S. Mruthyunjaya, “Closed-Form Dynamic Equations of the General Stewart Platform through the Newton–Euler Approach,” *Mech. Mach. Theory*, vol. 33, no. 7, pp. 993–1012, 1998.

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