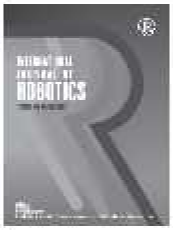

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Control of Quadrotor Using Sliding Mode Disturbance Observer and Nonlinear H_∞

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A B S T R A C T

In this paper, a nonlinear model of the underactuated six degrees of freedom (6 DOF) quadrotor helicopter was derived based on the Newton-Euler formalism. A new nonlinear robust control strategy was proposed to solve the stabilizing and path following problems in presence of external disturbances and parametric uncertainties. The proposed control structure consist of a sliding mode control based on disturbance observer (SMDO) to track the reference trajectory together with a nonlinear H_∞ controller to stabilize the rotational movements. Simulation results in the presence of aerodynamic disturbances and parametric uncertainties are presented to corroborate the effectiveness and the robustness of the proposed strategy.

1. Introduction

Recently unmanned aerial vehicles (UAVs) have wide area of applications either for military or civil purposes. Control of these systems has become popular due to their usefulness in rescue, surveillance, inspection, mapping, etc. UAV flight control system should make these performance requirements achievable by improving tracking and disturbance rejection capability. So, robustness is one of the critical issues which must be considered in the control system design for such high-performance autonomous UAV. Quadrotor is one of the most preferred types of small unmanned aerial vehicles. It is capable of vertical take-off and landing (VTOL), but it does not require complex mechanical linkages, such as swash plates that commonly appear in typical helicopters [1].

To achieve robustness and guarantee the stability of system, robust control strategies have been applied and investigated by many researchers. In [2] a feedback linearization-based controller with a high order sliding mode observer running parallel, is applied to a quadrotor unmanned aerial vehicle in presence of parameter uncertainties and external disturbances. A sliding mode disturbance observer was presented in [3]; this controller yields continuous control that is robust to the bounded disturbances and uncertainties. There are some limitations in the range of uncertainty and noise power which can be handles by sliding mode controller; so quadrotor helicopter has also been controlled using H_∞ controller. In [4] a mixed robust feedback linearization with linear GH_∞ controller is applied to a nonlinear quadrotor, the results demonstrated that the overall system was robust to uncertainties in system parameters and disturbances when weighting functions are chosen properly. In [5] control law based on back stepping for translational movements and nonlinear H_∞ controller to stabilize the rotational motion are combined to perform path following in the presence of external disturbances and parametric uncertainties; this strategy is only able to reject sustained disturbances applied to the rotational motion.

In the present paper, Tailor series expansion will be used to solve Hamilton-Jacobi-Isaacs Partial Differential Equation (HJI PDE) problem in nonlinear H_∞ control design. A major problem with this polynomial approximation approach is that even for low order systems, the controller expression becomes very complicated [6]; so the main idea in this paper, is to combine the advantages of the SMDO control methodology, for translational movements, with the capacity of the nonlinear H_∞ theory to stabilize the rotational motion in the presence of disturbance and uncertainty.

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The remainder of this paper is then organized as follows. fIGThe control strategy is exposed in Section III; in this section, two approaches of nonlinear robust control design are proposed: the SMDO control for translational movements and the nonlinear H_∞ control for the rotational subsystem. Section IV is devoted to simulation results.

2. Quadrotor Dynamics

In this section, model of the quadrotor rotorcraft is presented. Quadrotor platform has the shape of the cross (+) with a motor and propeller placed on each one of the four ends of the axes. Two propellers rotate counter clockwise and the others rotate clockwise such that the total torque of the system is balanced (approximately canceling gyroscopic effects and aerodynamic torques in stationary trimmed flight). Vertical motions of the quadrotor are generated by varying the rotor speeds of all four motors. The helicopter tilts towards the direction with lower lift rotor and accelerates along that direction. The basic model of an unmanned quadrotor is shown in Fig. 1. Pitch movement is obtained by increasing (reducing) the speed of the rear motor and reducing (increasing) the speed of the front motor. The roll movement is obtained similarly using the lateral motors. The yaw movement is obtained by increasing (decreasing) the speed of the front and rear motors and decreasing (increasing) the speed of the lateral motors (Fig. 2).

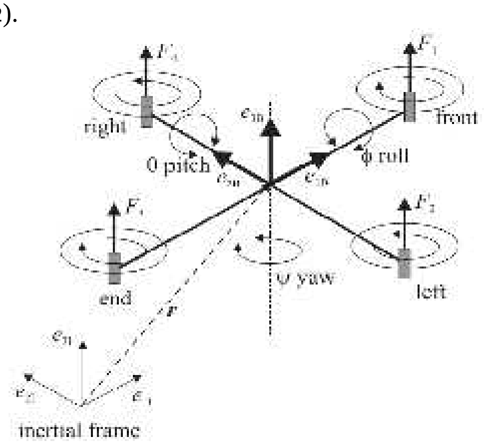

Fig. 1: The quadrotor helicopter configuration with roll-pitch-yaw Euler angles.

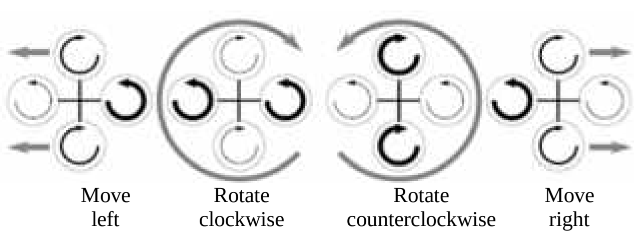

Fig. 2: The quadrotor motion description, the arrow width is proportional to propeller rotational speed [14].

The dynamics of the four rotors are relatively faster than the main system and thus it can be neglected here. . Earth fixed frame and body fixed frame are represented by $E=\{e_{1I},e_{2I},e_{3I}\}$ and $B=\{e_{1B},e_{2B},e_{3B}\}$, respectively. The generalized coordinates of the rotorcraft are $q=(x,y,z,\varphi,\theta,\psi)$, where (x,y,z) represents the absolute mass center position of the quadrotor with respect to an internal frame. (φ,θ,ψ) are the three Euler angles $(-\pi/2 \le \phi < \pi/2; -\pi/2 \le \theta < \pi/2; -\pi \le \psi < \pi)$ representing the orientation of the rotorcraft, namely roll-pitch-yaw of the vehicle. The translational and rotational coordinates are assumed in the form $\zeta=(x,y,z)^T \in R^3$ and $\eta=(\varphi,\theta,\psi) \in S^3$.

Considering the quadrotor as a single rigid body with 6 DOFs and neglecting the ground effect, the equation of motion for the rigid body is obtained in Newton-Euler formalism. The equations of motion for the helicopter can be written as follow [7]:

$$\begin{bmatrix} m I_{3\times3} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{V} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} \omega \times mV \\ \omega \times I\omega \end{bmatrix} = \begin{bmatrix} F_{ext} \\ \Gamma_{ext} \end{bmatrix} \quad (1)$$

Where m is the body total mass. V is the velosity vector in body axes, ω is angular velocity in the body axes and $I=diag(I_{xx},I_{yy},I_{zz})$ is the body inertial matrix. The external forces and moments are expressed in the body- fixed frame as

$$\begin{cases} F_{ext} = F_{prop} - F_{aero} - F_{grav} \\ \Gamma_{ext} = \Gamma_{prop} - \Gamma_{aero} \end{cases} \quad (2)$$

where F_{grav} is the gravity force with G=col (0;0,9:81) m/s^2 $F_{aero} = col(A_u, A_v, A_w)$ and $\Gamma_{aero} = col(A_p, A_q, A_r)$ with aerodynamic functions, $A_i = 1/2\,\rho\,C_i W^2$, are the aerodynamic forces and moments acting on the UAV respectively. C_i is aerodynamic coefficients, ρ is the air density, and $W = \Omega - \Omega_{air}$ is the velocity of the aircraft with respect to the air [8]. The main forces and moments acting on quadrotor are those produced by the propellers, F_{prop} and T_{prop}. These aerodynamic forces and moments are derived using the combination of momentum and blade element theory as $F_i = 1/2\,\rho A C_T (\Omega_i r)^2$; where A is the blade area, ρ the air density, r is radius of the blade and Ω the angular velocity of the propeller and C_T is aerodynamic coefficient [9]. At hover and semi hover, the thrust and drag forces are proportional to the square of the rotation speed of propellers, thus the thrust and drag forces, as in [10], are given by $F_i \approx K_T \Omega_i^2$ and $D_i \approx K_D \Omega_i^2$; Where K_T and K_D are constant and Ω_i

is the propeller's rotation speed. The system is subjected to one main force (thrust) F^b and three body torques (τ_φ, τ_θ, τ_ψ) which represent the control inputs to quadrotor and can be defined as follow:
vertical force
$U_1 = f_{prop} = K_T \sum_{i=1}^{4} \Omega_i^2$; moment control inputs, $\tau_{prop} = (\tau_\varphi, \tau_\theta, \tau_\psi)$ where $\tau_\varphi = U_2 = K_T(\Omega_1^2 - \Omega_3^2)$ is roll moment input, $\tau_\theta = U_3 = K_T(\Omega_2^2 - \Omega_4^2)$ is pitch moment input and $\tau_\psi = U_4 = K_D(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2)$ is yaw moment input. To derive equation of motion, the relation between body velocity vector, (V, ω), and inertial velocity vector, $(\dot{\xi}, \dot{\eta})$, is needed (kinematic equation):

$$\begin{cases} \dot{\xi} = R_t V \\ \omega = R_r \dot{\eta} \end{cases} \tag{3}$$

swhere R_t and R_r are translational and rotational velocity matrices between body frame and inertial frame respectively, such that

$$R_r = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\varphi & c\theta s\varphi \\ 0 & -s\varphi & c\varphi c\theta \end{bmatrix}$$
$$R_t = \begin{bmatrix} c\varphi c\psi & s\varphi s\theta c\psi - c\varphi s\psi & c\varphi s\theta c\psi + s\varphi s\psi \\ c\theta s\psi & s\varphi s\theta s\psi + c\varphi c\psi & c\varphi s\theta s\psi - s\varphi c\psi \\ -s\varphi & s\varphi c\theta & c\varphi c\theta \end{bmatrix} \tag{4}$$

s(.) and c (.) is sine and cosine of Euler angel.
Similarly, the forces and moments can be transformed based on R_t between the coordinate systems. In the body frame the forces are defined as

$$F_B = \begin{bmatrix} F_{xB} \\ F_{yB} \\ F_{zB} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sum_1^4 F_i \end{bmatrix}$$
$$F_E = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = R_t F_B = (\sum_1^4 F_k) \begin{bmatrix} c\varphi s\theta c\psi + s\varphi s\psi \\ c\varphi s\theta s\psi - s\varphi c\psi \\ c\varphi c\theta \end{bmatrix}$$

Therefore equation of motion in the earth frame are represented as

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x - A_u \\ F_y - A_v \\ F_z - A_w \end{bmatrix} \tag{5}$$

From (1) and (5), the forces equation of quadrotor expressed in inertial frame are modeled by the following equations [9]:

$$\begin{cases} \ddot{X} = 1/m\,((\cos\varphi \sin\theta \cos\psi + \sin\varphi \sin\psi)U_1 - A_u) \\ \ddot{Y} = 1/m\,((\cos\varphi \sin\theta \sin\psi - \sin\varphi \cos\psi)U_1 - A_v) \\ \ddot{Z} = -g + 1/m\,((\cos\varphi \cos\theta)U_1 - A_w) \end{cases}$$

Where X, Y are, the coordinates in the horizontal plane and Z is vertical position. Assuming small angles φ, θ and ψ, the rotational equations, can be simplified to be as follows

$$\begin{cases} \ddot{\varphi} = 1/I_{xx}\,(\dot{\theta}\dot{\psi}(I_{yy} - I_{zz}) + lU_2 - A_p) \\ \ddot{\theta} = 1/I_{yy}\,(\dot{\varphi}\dot{\psi}(I_{zz} - I_{xx}) + lU_3 - A_q) \\ \ddot{\psi} = 1/I_{zz}\,(\dot{\theta}\dot{\varphi}(I_{xx} - I_{yy}) + U_4 - A_r) \end{cases}$$

By choosing state vector as
$X = [x, \dot{x}, y, \dot{y}, z, \dot{z}, \varphi, \dot{\varphi}, \theta, \dot{\theta}, \psi, \dot{\psi}]^T$
and control input vector as
$U = [U_1, U_2, U_3, U_4]^T = [F_{aero}, \tau_\varphi, \tau_\theta, \tau_\psi]^T$
the equation of motion in state space is $\dot{X} = f(X, U)$, where

$$f(X,U) = \begin{pmatrix} \dot{x} \\ 1/m\,((\cos\varphi \sin\theta \cos\psi + \sin\varphi \sin\psi)U_1 - A_u) \\ \dot{y} \\ 1/m\,((\cos\varphi \sin\theta \sin\psi - \sin\varphi \cos\psi)U_1 - A_v) \\ \dot{z} \\ -g + 1/m\,((\cos\varphi \cos\theta)U_1 - A_w) \\ \dot{\varphi} \\ 1/I_{xx}[(\dot{\theta}\dot{\psi}(I_{yy} - I_{zz}) + lU_2 - A_p)] \\ \dot{\theta} \\ 1/I_{yy}[(\dot{\varphi}\dot{\psi}(I_{zz} - I_{xx}) + lU_3 - A_q)] \\ \dot{\psi} \\ 1/I_{zz}[(\dot{\theta}\dot{\varphi}(I_{xx} - I_{yy}) + U_4 - A_r)] \end{pmatrix} \tag{6}$$

In this model the Euler angles and their time derivatives do not depend on translation components. On the other hand, translations depend on the angles; so one can ideally imagine the overall system described by (5), is being composed of two subsystems: the angular rotations (that is fully actuated system, 3 inputs and 3 outputs) and the linear translations (that is under actuated system with only one control input).

3. Control

To achieve robust path following, two techniques, capable of controlling the helicopter in presence of external disturbances and parametric uncertainties are combined. The proposed control structure is a hierarchical scheme consisting of a sliding mode control based on disturbance observer controller (SMDO) to track the reference trajectory together with a nonlinear H_∞ controller to stabilize the rotational movements. In the case of the quadrotor, the three space positions and the yaw angle were chosen for trajectory tracking. Dynamic equation of the quadrotor shows that it is impossible to control the translational subsystem directly by the motor torque (due to the underactuated nature), therefore at the first, the trajectory block generates the desired trajectory of the vehicle in terms of space position and yaw angle. Thereby, starting from a reference trajectory for the translational subsystems, x_c, y_c, z_c, and their derivatives, by using the SMDO control law, the virtual control u_1, is computed. This control signal is composed of the command of the total thrust produced by the propellers, U_1, and the command rotation matrix, R_c, a function of the yaw angle. Therefore, this control cannot be used directly as a command for the rotational subsystem. By using simple mathematical analysis, reference Euler angles (ϕ_c, θ_c) computed. Finally a nonlinear H_∞ controller for rotational subsystem is used to track these reference signals to follow the desired x-y movement. The overall scheme of the control strategy is depicted in Fig. 3.

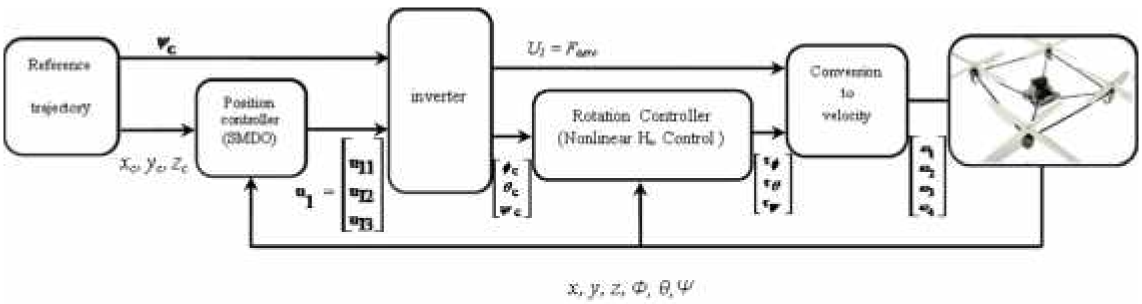

Fig. 3: UAV control scheme

3.1. Sliding mode disturbance observer

Sliding mode control (SMC) is a special nonlinear control strategy which has strong robustness against disturbances and uncertainty as well as guarantee the stability of a system [11]. However it presents the inconvenience of control signal high frequency switching which is not appropriate for UAV's control system. Robust sliding mode control using sliding mode disturbance observer allow us to consider unmanned rotorcrafts with bounded, continuous output tracking for translational subsystem. In this technique, at first, disturbance observer estimated and compensated the disturbance so switching gain in SMC decrease and chattering weakens. In the second step, the rest of the disturbance is canceled by the sliding mode control, and as a result, a very robust system is obtained, without the use of an additional sensor and even at the relatively low sampling frequency [12].

The output tracking problems focus on systems such as the following

$$\begin{cases} \dot{x} = f(x,t) + G(x,t)u \\ y = h(x,t) \end{cases}$$

Where $x \in \mathbb{R}^n, f(x,t) \in \mathbb{R}^n, y \in \mathbb{R}^m, u \in \mathbb{R}^m$ and $h(x,t) = [h_1, ..., h_m]^T \in \mathbb{R}^m$, $g(x,t) = [g_1, ..., g_m]^T \in \mathbb{R}^{n \times m}$, are analytic vector and matrix functions. Assuming the system is completely feedback linearizable in a reasonable compact domain $x \in \Gamma(x)$, the system can be transformed to the regular format

$$\begin{bmatrix} y_1^{(r_1)} \\ y_2^{(r_2)} \\ \cdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_f^{(r_1)} h_1(x,t) \\ L_f^{(r_2)} h_2(x,t) \\ \cdots \\ L_f^{(r_m)} h_m(x,t) \end{bmatrix} + E(x,t)u \qquad (7)$$

with

$$E(x,t) = \begin{bmatrix} L_{g_1}(L_f^{r_1-1}h_1) & L_{g_2}(L_f^{r_1-1}h_1) & \ldots & L_{g_m}(L_f^{r_1-1}h_1) \\ L_{g_1}(L_f^{r_2-1}h_2) & L_{g_2}(L_f^{r_2-1}h_2) & \ldots & L_{g_m}(L_f^{r_2-1}h_2) \\ \ldots & \ldots & \ldots & \ldots \\ L_{g_1}(L_f^{r_m-1}h_m) & L_{g_2}(L_f^{r_m-1}h_m) & \ldots & L_{g_m}(L_f^{r_m-1}h_m) \end{bmatrix}$$

where $L_f^{r_i-1}h_i$ and $L_{g_i}(L_f^{r_i-1}h_i)$, $\forall i = 1, ..., m$ are corresponding Lie derivatives $|E(x,t)| \neq 0, \forall x \in \Gamma$ and $\bar{r} = [r_1, r_2, ..., r_m]^T$ is a vector relative degree [13]. Here, our goal is to compute a control u such that, derive the output of the system, y, to track the reference trajectory y_c. Introducing the tracking error as $e = y_c - y$, and sliding variable σ as

$$\sigma_i = e_i^{(r_i-1)} + c_{i,r_{i-2}} e_i^{(r_i-2)} + \ldots + c_{i,1} e_i^{(1)} + c_{i,0} e_i \qquad (8)$$

where $c_{i,k} \succ 0 \in \mathbb{R}$ and $\pm = [\pm_1, \pm_2, ..., \pm_m]^T$, $e_i = y_{i,c} - y_i$, $e_i^{(k)} = d^k e_i / dt^k$ In (6), introducing $\tilde{u}_i = (Eu_i)$, we have

$$e_i^{(r_i)} = y_{i,c}^{(r_i)} - L_f^{r_i} h_i(x,t) - \tilde{u}_i \qquad (9)$$

Derivative from (7) and substitute in to equation (8), we obtain

$$\dot{\pm}_i = y_{i,c}^{(r_i)} - L_f^{r_i} h_i(x,t) - \tilde{u}_i + c_{i,r_{i-2}} e_i^{(r_i-1)} + ... + c_{i,1} e_i^{(2)} + c_{i,0} e_i^{(1)} \qquad (10)$$

By defining

$$\cdot_i = y_{i,c}^{(r_i)} - L_f^{r_i} h_i(x,t) + c_{i,r_{i-2}} e_i^{(r_i-1)} + ... \qquad (11)$$
$$\cdot_i = \cdot_i^0 + \Delta \cdot_i$$

where $\cdot_i^0$ is the known part of $\cdot_i$ while $\Delta \cdot_i$ represents the bounded uncertainties ($\|\Delta \cdot_i\| \le L_i$) acting on the system. By substituting equation (10) into (9) we have

$$\dot{\pm}_i = \cdot_i^0 + \Delta \cdot_i - \tilde{u}_i \qquad (12)$$

The term $\cdot_i^0$ is known; therefore, we can compensate for it directly by $\tilde{u}_{i,0} = \cdot_i^0$. Consequently, defining the control input as $\tilde{u}_i = \tilde{u}_{i,0} + \tilde{u}_{i,1}$ and substituting in to equation (12), we obtain

$$\dot{\pm}_i = \Delta \cdot_i - \tilde{u}_{i,1} \qquad (13)$$

Equation (13) shows that if the control with deriving [illegible] to zero could be designed, this control would then be equal to the disturbance term. At first, we need to introduce the auxiliary sliding variables s_i and z_i such that

$$\begin{cases} s_i = \pm_i + z_i \\ \dot{z}_i = \tilde{u}_{i,1} - v_i \end{cases} \qquad (14)$$

From (14) and equation (13), we obtain

$$\dot{s}_i = (\Delta \cdot_i - \tilde{u}_{i,1}) + (\tilde{u}_{i,1} - v_i)$$
$$\dot{s}_i = \Delta \cdot_i - v_i \qquad (15)$$

It can be easily shown that the control that will derive s_i to zero and maintain it there for all subsequent time is $v_i = (°_i + L_i) sign(s_i)$. This control displays high-frequency switching that can be filtered by a low pass filter such as

$\hat{v}_{i,eq} = 1/(1 + \ _i s) v_i$

The choice of time constant, β_i, is important; very big time constant results in a time lag between the disturbance and its estimation, and too small time constant results in rendering the control ineffective. Moreover the constant β also need to be larger than Δt, the sampling time of the sensor and actuators. Both these considerations can be summarized as

$$\lim_{\to 0} \frac{\Delta t}{} = 0 .$$

From equation (15), one can see that if the auxiliary sliding variable s_i stabilizes at zero, then $\Delta \cdot_i - v_i = 0$. As a result, the control $v_{eq,i}$, would be an exact estimation of $\Delta \cdot_i$, so we have designed a sliding mode disturbance observer where equivalent control $\hat{v}_{i,eq}$ is, in fact, an estimation of the unknown disturbance term $\Delta \cdot_i$.

The resultant control

$$\begin{cases} \tilde{u}_i = \tilde{u}_{i,0} + \tilde{v}_{i,eq} \\ \tilde{u}_{i,0} = \cdot_i^0 \end{cases}$$

Completely compensates for the system dynamics, but a final control term needs to be added to drive [illegible] to zero and satisfy the sliding mode existence condition. Therefore,

$$\tilde{u}_i = \tilde{u}_{i,0} + \hat{v}_{i,eq} + K_{0,i} \pm_i \qquad (16)$$

where $K_{i,0} > 0$ is chosen to ensure fast reaching of the sliding surface by σ as soon as the disturbance is estimated and compensated (i.e., the convergence of σ_i cannot be faster than that of s_i). Substituting equation (16) into (12) yields $\dot{\pm}_i = -K_{0,i} \pm_i$, which satisfies the sliding mode existence condition $\dot{\pm}_i \pm_i < 0$. Therefore final smdo/smc control signal is $u = E^{-1}[\cdot_0^i + \hat{v}_{eq} + K_0 \pm]$.

3.2. Application on system

By rewriting the equation of motion (6) As

$$\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{z} = w \\ \dot{u} = 1/m\,((\cos ¨ \sin ¯ \cos \cdot + \sin ¨ \sin \cdot) U_1 - K_{tx} u) + d_u \\ \dot{v} = 1/m\,((\cos ¨ \sin ¯ \sin \cdot - \sin ¨ \cos \cdot) U_1 - K_{ty} u) + d_v \\ \dot{w} = -g + 1/m\,((\cos ¨ \cos ¯) U_1 - K_{tz} w) + d_w \end{cases}$$

$$\Rightarrow \begin{cases} \dot{X}_1 = X_2 \\ \dot{X}_2 = f(X_2) + u_1 + d \end{cases}$$

Where X_1, X_2 represent, respectively, the position and linear velocity V of the quadrotor in the inertial frame; $f(.)$ denotes the known parts of the dynamics and d represents the model uncertainties and external disturbances acting on the system. This term is unknown but norm bounded on a reasonable flight domain. Finally virtual control u_1 is force vector

$$u_1 = \begin{bmatrix} \frac{U_1}{m}(c¨\, s¯\, c· \; + s¨\, s·\,) \\ \frac{U_1}{m}(c¨\, s¯\, s· \; - s¨\, c·\,) \\ -g + \frac{U_1}{m}(c¨\;\, c¯\,) \end{bmatrix}$$

By defining tracking error between the position command profile and the actual position as e = [x_c-x, y_c-y, z_c-z]^T and sliding variable $±_i = [±_x, ±_y, ±_z]^T$ as $±_{1,i} = e_{1,i} + C^1_{i,0} e_{1,i}$, where the gain $C^1_{i,0}$ is chosen such that the tracking error, *e*, exhibit the desired behavior in the sliding mode; the final SMC/SMDO control signal is

$$\begin{cases} ±_{1,i} = \dot{e}_{1,i} + 1.25 e_{1,i} \\ s_{1,i} = ±_{1,i} + z_{1,i} \\ \dot{z}_{1,i} = \tilde{u}_{1,i} - v_{1,i} \\ v_{1,i} = (°_{1,i} + L_{1,i})\,\mathrm{sgn}(s_{1,i}) \\ \hat{v}_{eq,1,i} = \dfrac{1}{(0.01s+1)^2} v_{1,i} \\ u_{1,i} = ·\,^{1}_{0,i} + \hat{v}_{eq,1,i} + 15 ±_{1,i} \end{cases}$$

To compute thrust and the Euler angels commands, the control signal, $u_1 = [u_{1,1}, u_{1,2}, u_{1,3}]$, needs to be inverted. By using simple mathematical analysis, reference Euler angles $(¨_c, ¯_c)$ computed as bellow

$$\begin{cases} U_1 = \left\| m(u_{1,1} + G) \right\| \\ ¨_c = a\sin\left(\dfrac{m}{U_1}(s·_c u_{1,1} - c·_c u_{1,1})\right) \\ ¯_c = a\sin\left(\dfrac{\frac{m}{U_1} u_{1,1} - s·_c s¨_c}{c·_c c¨_c}\right) \\ ·_c = ·_c \end{cases}$$

Finally, a nonlinear H_∞ controller is used to perform the quadrotor helicopter stabilization.

3.3. Nonlinear H_∞ controller for stabilization

The nonlinear H_∞ control has great potential for handling uncertain nonlinear control problems. The goal of this control theory, presented by Van der Schaft in [14], is to achieve a bounded ratio between the energy of the so-called error signals and the energy of the disturbance signals. It has been shown that the solution of the nonlinear H_∞ control problem related to the solvability of a particular type of HJI equations or inequalities which replace the Riccati equations in the case of the linear H_∞ control formulation [15]. Hence, one of the major concerns in the nonlinear H_∞, control theory is the computation issue involving the solution of the HJI equations or inequalities.

In this section, a nonlinear H_∞ controller for the rotational subsystem is developed, by using Taylor series expansion to solve HJI PDEs.

We consider the following multi input, multi output nonlinear system equations

$$\dot{x} = f(x) + g_1(x)w + g_2(x)u \qquad (17)$$

$$z = h_1(x) + k_{12}(x) \qquad (18)$$

Where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^{m_2}$ is the plant input, $w \in \mathbb{R}^{m_1}$ is a set of exogenous input variable and $z \in \mathbb{R}^p$ is penalty variable. It is assumed that all functions involved in this setup are smooth and defined in a neighborhood *U* of the origin in $\mathbb{R}^n$ and have zero values at the origin. The nonlinear H_∞ state feedback problem control takes the form:

$$u = l(x) \qquad (19)$$

In which $l : U \to \mathbb{R}^{m_2}$ is a C^k function (for some $k \geq 1$ satisfying $l(0) = 0$) The purpose of control is twofold: to achieve closed loop stability and to attenuate the effect of the disturbance input *w* to the penalty variable *z*. Here closed loop system is stable and the disturbance attenuation is characterized in the following way. Given a real number $¥ > 0$, it is said that the exogenous signals are locally attenuated by γ if there exists a neighborhood, *U* of the point $x = 0$ such that for every $T > 0$ and for every piecewise continuous function $w : [0,T] \to \mathbb{R}^{m_1}$ for which the state trajectory of the closed loop system starting from $x(0) = 0$ remains in *U* for all $t \in [0,T]$, the response $z : [0,T] \to \mathbb{R}^p$ and (19) satisfies

$$\int_0^T z^T(s)Z(s)ds \leq ¥^2 \int_0^T w^T(s)w(s)ds \qquad (20)$$

The Hamiltonian associated with the above problem is defined as follows

$$H(x,p,w,u) = p^T(f(x) + g_1(x)w + g_2(x)u) + \tfrac{1}{2}(\|h_1(x) + k_{12}(x)\|^2 - ¥^2\|x\|^2)$$

It was shown in [16], that if there exists a smooth positive definite C^1 function $V(x)$, locally defined in a neighborhood of the origin in $\mathbb{R}^n$ that satisfies

$$H(x, V_x^T) \overset{def}{=} H(x, V_x^T, \sim_1(x, V_x^T), \sim_2(x, V_x^T)) \quad (21)$$

Or more explicitly,

$$V_x f(x) + \frac{1}{2}h_1^T h_1 + \frac{1}{2}V_x\left(\frac{g_1 g_1^T}{¥^2} - g_2 R^{-1} g_2\right)V_x^T \quad (22)$$
$$= 0$$

Where V_x denotes the Jacobian matrix of $V(x)$; then the state feedback control law given by

$$u = -R^{-1}(x)g_2^T(x)V_x^T \quad (23)$$

achieves disturbance attenuation with performance level specified by γ.

Due to the nonlinear nature, it is rarely possible to find a closed form solution for the HJI equation. An approximation approach to solve (22) via the Taylor series was suggested in [16].The advantage of using this method is that it is easy purely algebraic.

3.4. Application on system

The penalty variable, z, comprises any output whose L_2 norm is desired to be minimized. This variable can include tracking error, actuator effort and frequency response criteria. In the nonlinear H_∞ controller design, most static weighting functions have been explored. In this paper by defining tracking error as $e = (\phi - \phi_c, \theta - \theta_c, \psi - \psi_c)$, the penalty variable z, was chosen as a dynamic weighting function as bellow.

$z = [z_1, z_2]^T$, where $z_1 = \frac{1}{s}(k_1 e + k_2 \dot{e})$ and $z_2 = Wu$.

4. Simulation Result

For simulation, model os-4 quadrotor in [17] was used. The simulations performed in Matlab®/SIMULINK® environment, include Hover and sinusoid and circle trajectory that robustness of the controller was tested by applying external disturbance and parametric uncertainty and sensor noise.

4.1. Hover mode

The first simulation tests the controller's ability to reach and maintain hover mode. This test is simulated under the conditions are summarized in Table 1.

Table 1. Initial conditions for Hover mode

Initial condition	Constant position (Hover)	Disturbance	Uncertainty
$(x_0, y_0, z_0) = (0,0,0)^m$ $(u_0, v_0, w_0) = (0,0,0)^{m/s}$ $(\phi_0, \theta_0, \psi_0) = (0,0,0)^{rad}$ $(p_0, q_0, r_0) = (0,0,0)^{rad/s}$	$(x_c, y_c, z_c) = (0,1,2)^m$ $\psi_c = \frac{\pi}{4} = 0.78\,rad$	None	None

This simulation without uncertainty or disturbance is used to evaluate the accuracy of the controller in ideal conditions. From Fig. 4 it can be seen that controller is able to reach the desired position quickly and maintain stable hover afterwards.

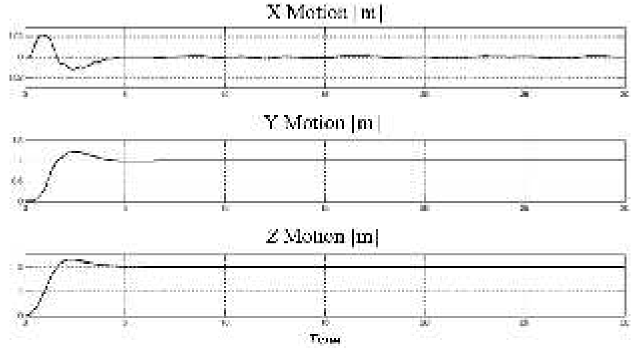

Fig. 4: Position in hover mode

Fig. 5 also confirms the accuracy of the control as each Euler angle follows the reference trajectory.

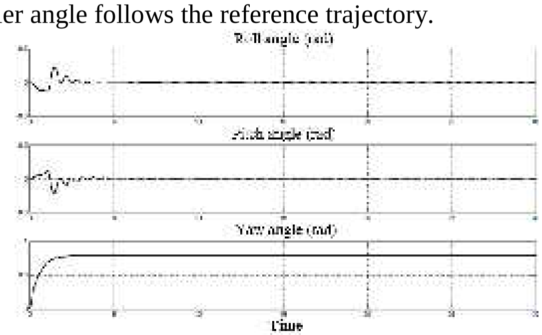

Fig. 5: Euler angles in hover mode

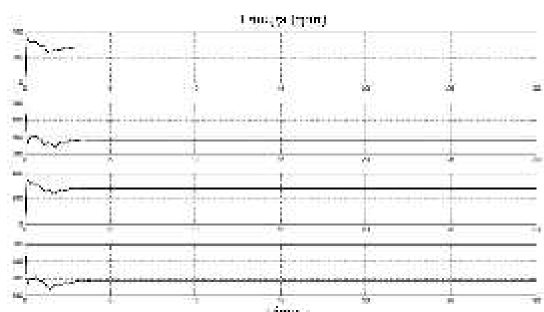

Fig. 6: Speed control inputs of four rotors in hover mode

Fig. 6 shows the speed control signals of four rotors to achieve hover mode. These control signals are continues and have a small amplitude, that is applicable on motors.

4.2. Sinusoidal trajectory with disturbance and parametric uncertainty

The accuracy of controller having been established by previous test. The purpose of the following simulations is to assess the controller's robustness. This simulation is performed on a sinusoidal trajectory along the x axis of amplitude 2^m executed at constant altitude $z=2^m$ with 30% parametric uncertainty and collision disturbance applied as a downward force on the side of the vehicle (a torque is also generated). Initial conditions in this test are expressed in Table 2.

Table 2: initial conditions for sinusoid trajectory

Initial condition	Sinusoidal Trajectory	Uncertainty	Disturbance
$(x_0, y_0, z_0) = (0,0,0.5)^m$ $(u_0, v_0, w_0) = (0,0,0)^{m/s}$ $(¨_0, ¯_0, ·_0) = (0,0,0.5)^{rad}$ $(p_0, q_0, r_0) = (0,0,0)^{rad/s}$	$x_c = 2\cos(0.1t)$ $·_c = ®/3$	30%	$F_{shock} = [0,0,-8]$ $²_{shock} = [9,4,0]$ $In\ t = 20^{sec}$

By Applying 30% uncertainty in parameter of quadrotor such as mass, aerodynamical coefficient, inertial element and …, by applying a shock 20 second after the beginning of the test, as can be seen from Fig. 7, and Fig. 8 the controller is able to return quadrotor to reference trajectory in presence of disturbance and uncertainties.

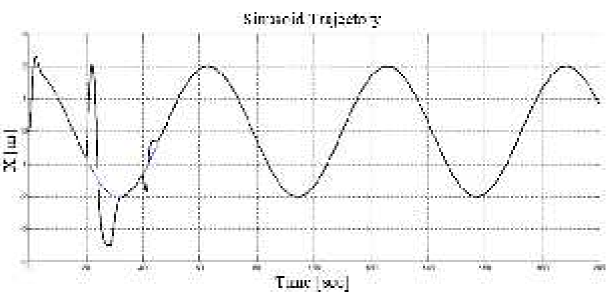

Fig. 7: Sinusoid trajectory along x axis with colision disturbance and parametric uncertainties

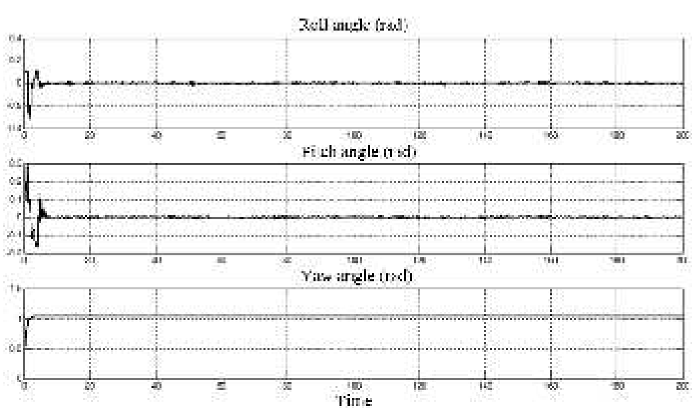

Fig. 8: Euler angles with colision disturbance and parametric uncertainties.

Fig. 9 shows the speed control signals and speeds of four rotors in this mode. These control signals are continues and have a small amplitude, that is applicable on motors.

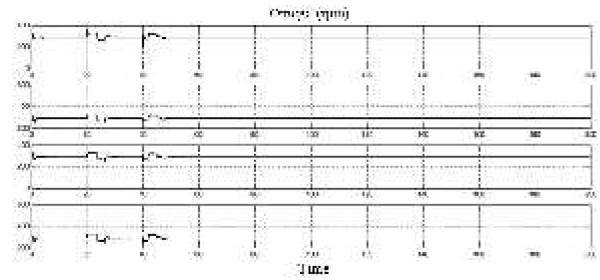

Fig. 9: Speed control inputs of four rotors with colision disturbance and parametric uncertainties.

4.3. Circle trajectory with parametric uncertainty and sensor noise

Initial conditions for third simulation are expressed in Table 3.

Table 3: Initial conditions for circle trajectory

Initial condition	Circle trajectory	Uncertainty	Sensor noise
$(x_0, y_0, z_0) = (0,0,0.5)^m$ $(u_0, v_0, w_0) = (0,0,0)^{m/s}$ $(¨_0, ¯_0, ·_0) = (0.2,0,0)^{rad}$ $(p_0, q_0, r_0) = (0,0,0)^{rad/s}$	$x_c = 4\cos(0.1t)$ $y_c = 4\sin(0.1t)$ $z_c = 5$ $·_c = 0.1$	20%	Noise power =0.001

Fig. 10 shows that controller is able to return quadrotor to reference trajectory in presence of 20% uncertainty and sensor noise.

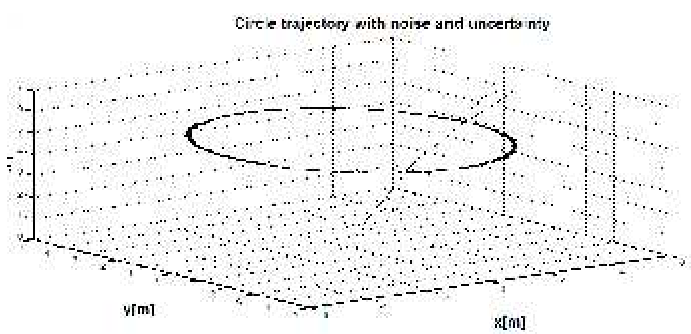

Fig. 10: Circle trajectory with parametric uncertainty and sensor noise.

5. Conclusion

To achieve a robust path following for the quadrotor helicopter, two robust techniques are combined. The proposed control structure consisting of a sliding mode control based on disturbance observer controller (SMDO) to track the reference trajectory and a nonlinear H_∞ controller based on a Taylor series expansion to stabilize the rotational movements. The results obtained during these simulations effectively proved the high robustness of the controller in the presence of collision disturbance, parametric uncertainty and sensor noise. In each case SMDO/NLH_∞ controller compute a robust continuous control able to compensate for the disturbance and maintain the vehicle close to its desired trajectory.

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