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Asymptotic Boundary Feedback Stabilization of a Flexible Gantry Manipulator with One Axis of Rotation

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ABSTRACT

This paper aims to develop a boundary control solution for a single-link gantry robot manipulator with one axis of rotation. The control procedure is considered with link's transverse vibrations while system undergoes rigid body nonlinear large rotation and translation. Initially, based on Hamilton principle, governing equations of hybrid motions as a set of partial differential equations (PDE) and ordinary differential equations (ODE) will be derived. The control objectives which are sought for include: moving the system to a desired position, regulating large angular position and finally suppressing the flexible link transverse vibrations simultaneously. By considering novel Lyapunov functions and avoiding any simplifications, In the presence of external boundary disturbance, proper control feedback signals and boundary disturbance observer are introduced in order to reach mentioned control objectives and compensate external boundary disturbance effect simultaneously. At last uniform ultimate boundedness of the closed loop system is proven in which by choosing proper design parameters, system states and position error converge exponentially to a small neighborhood of zero. In order to illustrate the performance of the proposed control method, numerical simulation results are provided.

1. Introduction

Development and motion control of flexible manipulators has received everlasting demand for robot fabrications and researchers (De Luca and Book, 2008). The most critical issues for these systems are dynamic accuracy, higher operating speed and ensuring the operating safety. Although employing lightweight links can improve the total efficiency, for high-speed precision transportation systems, the large amplitude undesirable vibrations due to the flexible properties of the manipulator and the external disturbances contribute to imprecise positioning which is time-consuming, extortionate and perilous as well. Therefore, the effective control methods for accurate positioning and vibration suppression are desirable and has attracted a great deal of attention currently and many publications have been devoted to Infinite Dimensional Control methods that guarantee the stabilization of infinite-dimensional Flexible structures (Halim and Cazzolato, 2006; Krstic, 2009; Luo et al., 2014). (Fang et al., 2003; Ismail et al., 2009; Vaughan et al., 2010).In which due to the representation of the flexible body as a distributed parameter system with infinite degrees of freedom, the complexity of control design rises.

Since PDE based stabilization producer has not been well developed compared to control approaches for ODE systems, control techniques such as assumed-mode, lumped-parameter, and finite element methods are used to simultaneous end effector regulation and vibration suppression, which all relay on discretizing the PDE model into a set of ODEs. Since control design based on over mention methods only consider a finite number of vibration modes, in order to avoid high order controlling problems and spillover observation and control occurrences, many new control approaches based on original PDE model without discretization for flexible links have been presented. Other approaches based on infinite-dimensional control procedure include distributed control methods, which is based on the implementation of a network of sensors and actuators distributed at interior points of the system. Enhancement of these kinds of devices such as piezoelectrics and strain gages leads to increase in cost and complexity.

Among infinite dimensional control methods, boundary control as comparison seems to be a more practical method. The implementations trait of these controllers makes them be widely applicable in many major control strategies for systems governed by PDEs (De Queiroz and Rahn, 2002). The BCs designed for the non-discretized PDE models ensure closed-loop stability for an infinite number of modes which avoid the spillover phenomena (Meirovitch and Baruh, 1983; Najafi et al., 2011; Vatankhah et al., 2015). Additionally, this method removes the in-domain sensing/actuating problem; on the other hand, the controllers basically admit apparent physical features. Therefore, several researchers have proposed applying the boundary controllers for a variety of flexible systems such as strings(Fung et al., 2002; He et al., 2012; Krstic, 2009), container cranes(He et al., 2017; He and Ge, 2016) flexible aircraft wings(He et al., 2018), satellites (W. He and Ge, 2015), composite plates (Najafi et al., 2010; Rastgoftar et al., 2010) and composite shells containing fluid (Najafi and Eghtesad, 2013).

Mainly, based on the benefits of the boundary control method, the researchers have been established this method to overcome the challenges facing the control of flexible manipulators.

For beams as a fundamental element in flexible manipulators, (Morgul, 1992) by using boundary control torque and force have developed a vibration control strategy. (Lotfazar et al., 2008) studied general in-plane trajectory tracking problem of a flexible beam using two-time scale control theory and BC method. Also, comprehensively (Ge et al., 2011) have developed control scheme for vibration suppression while dealing with compensation of the system parametric uncertainties and the disturbances uncertainties. Similarly, (Wei He and Ge, 2015) addressed vibration stabilization based on barrier Lyapunov function theory in order to consider boundary output constraint. Moreover, vibration regulation with input saturation is also considered in (He and Liu, 2019).

As mentioned before, based on BC technique novelty, this method has been extended for the different flexible manipulator's systems. (Queiroz et al., 1999) addressed asymptotic BC strategy for a nonlinear vertical flexible link by developing control torque on hub and control force on endpoint in order to regulate the vibration and drive the manipulator to the desired point.(Endo et al., 2009) consider two-one link flexible arm for grasping task and implement BC to achieve both asymptotic and exponential stability And expansively (He and Sun, 2016) have achieved the uniform boundedness of closed-loop system and existence and uniqueness of flexible vertical arm with input constraint . For nonlinear flexible arm (Tavasoli and Mohammadpour, 2018) asymptotically regulated a flexible robotic arm with 2-dimensional rigid body rotation. Moreover, (Liu et al., 2018) designed a controller for nonlinear 3-dimensional flexible arm ignoring gravitational energy.

Furthermore, in order to compensate unknown boundary disturbances, disturbance observers are implemented. (Kyung-Jinn Yang et al., 2005) have implemented disturbance observer of translating beam in which the value of the bonds of the disturbances are estimated. To compliment (He et al., 2013b) and (He, Ge, et al., 2015) investigated designing boundary disturbances in order to estimate the time-varying boundary disturbances.

In this paper, a novel BC strategy has addressed for gantry flexible manipulator systems. Proposed gantry manipulator systems are illustrated in figure 1. In this work, the transverse vibration and rotation angle have been regulated, meanwhile proposed manipulator has directed to the desired position. Moreover, boundary disturbance observer has been designed for the proposed system.

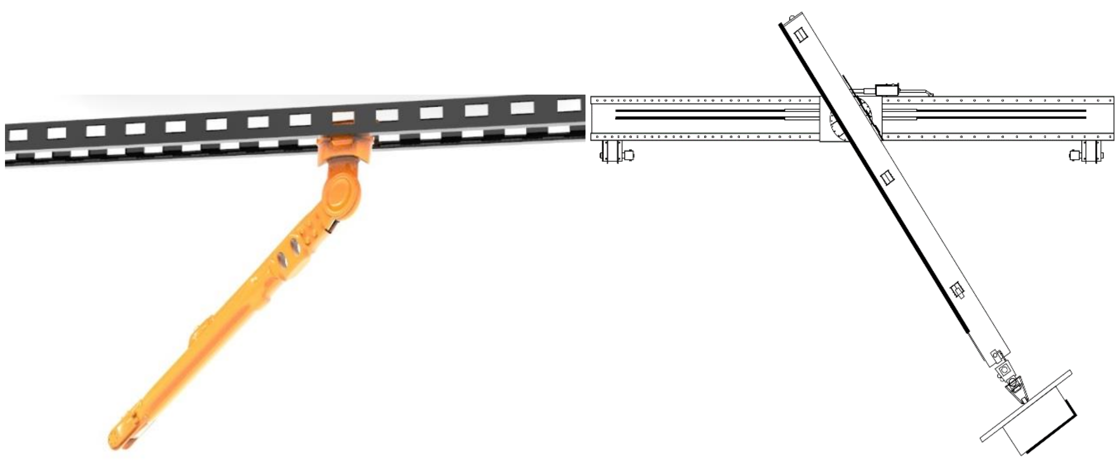

Figure 1. Proposed gantry manipulator systems

What has been considered in most of the above-aforementioned work; they have addressed the BC method based on linear flexible arms in the vertical plane. Furthermore, the designing control procedures were not relatively based on the original hybrid PDE-ODE dynamic modelling. To the best of our knowledge many nonlinearities, the effect of gravitational force and base position tracking to the desired position has been disregarded, which the ignorance of them affected the proposed controller performances. These assumptions may be appropriate for small angle regulation and slow motion of manipulator's base specifically near the set point. However, in order to achieve precise locating of the end effector, considering over mentioned issues are inevitable in many applications, especially in the vertical plane and in the presence of time-varying boundary disturbances. This effort is a comprehensive procedure for accurate boundary control of the gantry flexible manipulator system with transverse vibration and large rotational angle regulation. Moreover, it is worth to remark that, the control designing and stability procedures are based on original hybrid PDE-ODE proposed governing equations without any simplifications in order to overcome accurate positioning. Comparing with the existing studies, the main contributions of this research are summarized as follows:

i. Based on the Hamiltonian method, the original PDE-ODE equations of motion have been derived. The proposed dynamical model involved the nonlinearities and coupling effects between the large rotational angle dynamics, gravitational force, the flexibility, the payload and base mass dynamics and the varying boundary disturbances, which all exert considerable influence on control design strategy.

ii. A novel boundary control method for flexible gantry manipulator system by considering simultaneous vibrations suppression, large nonlinear angular rotation, base accurate positioning and avoiding any simplifications and spillover, have been proposed.

iii. For mention controlling purposes, an initiative nonlinear Lyapunov design approach has utilized in order to establish control rules which ensure the exponential stability of the closed-loop system without considering boundary disturbances. Additionally, uniform boundedness of the closed-loop system under the unknown time-varying boundary disturbance by adding boundary observer is achieved. Moreover, the boundedness of all closed-loop signals is also shown. These signals consist of the base's horizontal displacement, rotational angle, and boundary deflection and shear measurement with their derivatives which make them practical.

The rest of this paper is organized as follows. Section 2 introduces the system kinematics and dynamics. Section 3 is dedicated to some preliminaries. Section 4 consists of designing boundary control rules by utilizing the Lyapunov method in which a complex and proper Lyapunov functional will be adopted. Moreover, also based on over mentioned control inputs the exponential stability and uniform boundedness of the closed-loop system in the absence and presence of boundary disturbance have shown respectively. Section 5 includes simulation results, presented in order to show the effectiveness of the boundary control approach. Eventually, the conclusions are summarized in Section 6.

2. Mathematical Modeling

Figure 2 shows the schematic diagram of the proposed system consisting of a portable base, flexible arm, and payload at the bottom. Frame XOY is the fixed inertial frame, and the motion of the system takes place in a vertical plane.

Let t be the time, x the spatial coordinate along the longitude of flexible link, $w(x\ ,\ t)$ the transverse displacement due to lateral vibrations of the flexible link at the spatial coordinate x and time t, $\theta(t)$ is rotational angle and $\dot{\theta}(t)$, $\ddot{\theta}(t)$ are first and second order corresponding derivatives respect to time and $\eta(t)$ and $\dot{\eta}(t)$ refer to trolley position and velocity respectively . Moreover, let m_1 and m_2 represent the equivalent mass of base and payload respectively, l denotes the length of link, ρ the mass per unit of length and the subscripts x; t denotes the partial derivatives with respect to x; t, respectively.

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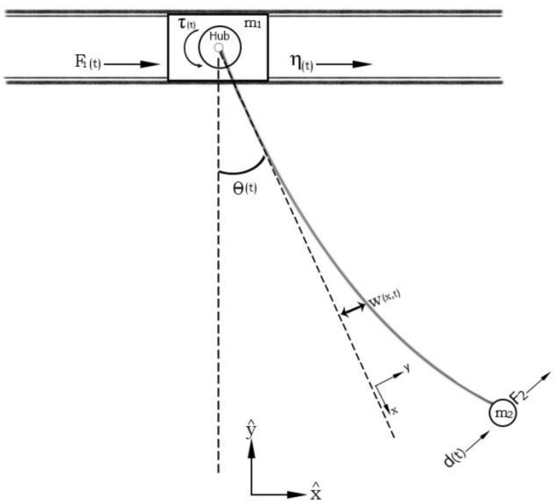

Fig 2. Schematic of flexible gantry manipulator with input forces and boundary disturbance

The kinetic energy of the Gantry flexible manipulator system can be represented as:

$$K_T = 1/2 m_1 \dot{\eta}^2 + 1/2\rho \int_0^{l(t)} \left(v_x{}^2 + v_y{}^2\right) dx + 1/2 m_2 \left(v_{xl}{}^2 + v_{yl}{}^2\right) \quad (1)$$

v_x and v_y denote the velocity vectors of flexible system in *x* and *y* direction and can be defined as:

$$v_x = \dot{\eta} + x\dot{\theta}\cos(\theta) + \left(w_t(x,t)\right)\cos(\theta) - w(x,t)\dot{\theta}\sin(\theta) \quad (2)$$

$$v_y = x\dot{\theta}(t)\sin(\theta(t)) + \left(w_t(x,t)\right)\sin(\theta(t)) - w(x,t)\dot{\theta}(t)\cos(\theta(t)) \quad (3)$$

Using equations (1) ,(2) and (3) the kinetic energy can be rewritten as:

$$\begin{aligned}
K_T &= 1/2 m_1 \dot{\eta}^2 \\
&+ 1/2\rho \int_0^l \left\{ \begin{array}{l} \dot{\eta}(t)^2 + \left(x\dot{\theta}(t)\right)^2 \\ + \left(w_t(x,t)\right)^2 \\ + \left(w(x,t)\dot{\theta}(t)\right)^2 \end{array} \right\} dx \\
&+ 1/2\rho \int_0^l \{ 2\dot{\eta}(t)\dot{\theta}(t) x\cos(\theta(t)) \\
&+ 2\dot{\eta}(t)\left(w_t(x,t)\right)\cos(\theta(t)) \\
&- 2\dot{\eta}(t)\dot{\theta}(t) w(x,t)\sin(\theta(t)) \\
&+ 2\left(w_t(x,t)\right)\dot{\theta}(t) x \} dx \\
&+ 1/2 m_2 \{ \dot{\eta}(t)^2 + \left(l\dot{\theta}(t)\right)^2 \\
&+ \left(w_t(l,t)\right)^2 + \left(w(x,t)\dot{\theta}\right)^2 \} \\
&+ 1/2 m_2 \{ 2\dot{\eta}(t)\dot{\theta}(t) l\cos(\theta(t)) \\
&+ 2\dot{\eta}(t) w_t(l,t)\cos(\theta(t)) \\
&- 2\dot{\eta}(t)\dot{\theta}(t) w(l,t)\sin(\theta(t)) \\
&+ 2 w_t(l,t)\dot{\theta}(t) l \}
\end{aligned} \quad (4)$$

The potential energy $E_p(t)$ due to the strain and gravity potential energy can be shown by:

$$E_p = E_{p1} + E_{p2} \quad (5)$$

In which

$$E_{p1} = 1/2 \int_0^l EI \left(w_{xx}(x,t)\right)^2 dx \quad (6)$$

$$E_{p1} = \rho g \int_0^l \left(\begin{array}{l} l - x\cos(\theta(t)) + \\ w(x,t)\sin(\theta(t)) \end{array} \right) dx + m_2 g \left(\begin{array}{l} l - l\cos(\theta(t)) \\ + w(l,t)\sin(\theta(t)) \end{array} \right) \quad (7)$$

In order to obtain the equations of motion, the extended Hamiltonian principle is applied.

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W_e\right) dt = 0 \quad (8)$$

$\delta W_e, \delta\eta, \delta w$ and $\delta\theta$ represent the virtual work, the virtual base displacement, virtual elastic transverse displacement and virtual rigid body angular rotation respectively.

The total virtual work done on the system due to external disturbance and control forces on trolley and payload is given by:

$$\delta w_e = (F_1(t))\delta\eta + (F_3(t) + d(t))\,\delta w(l,t) + (F_3(t)l + d(t)l + \tau(t))\,\delta\theta \quad (9)$$

Assumption 1. For unknown boundary disturbance $d(t)$, its derivative $\dot{d}(t)$ is bounded by positive constant $D \in \mathbb{R}^+$, such that $\left|\dot{d}(t)\right| \le D$, $\forall(t) \in [0,\infty)$.

$F_1(t)$ is the control force on the trolley respectively, $\tau(t)$ is the control torque exerted on the hub and similarly $F_3(t)$ and $d(t)$ are the control force and the external disturbance on the payload respectively and by integration by parts and some simplifications equation (8) can written as:

$$\begin{aligned}&\int_{t_2}^{t_1}(\sigma_1(x,t)\delta\theta)dt\\&+\int_{t_2}^{t_1}(\sigma_2(x,t)\delta w(x,t))dt\\&+\int_{t_2}^{t_1}(\sigma_3(x,t)\delta r)dt\\&+\int_{t_2}^{t_1}(\sigma_4(x,t)\delta w(l,t))dt\\&+\int_{t_2}^{t_1}(\sigma_5(x,t)\delta w(0,t))dt = 0\end{aligned} \quad (10)$$

Letting $\sigma_1(x,t)=0$, $\sigma_2(x,t)=0$, $\sigma_3(x,t)=0$ and $\sigma_5(x,t)=0$, the equations of motion and the boundary conditions can be obtained as following:

$$\begin{aligned}&\ddot{\theta}(t)\left(m_2 l^2 + \rho\, l^3/3\right) +\\&\left(\rho\, l^2/2 + m_2 l\right)\ddot{\eta}(t)\cos(\theta(t))\\&+\left(\rho\, l^2/2 + m_2 l\right)g\sin(\theta(t))\\&+\int_0^l \rho\begin{bmatrix}2w(x,t)(w_t(x,t))\dot{\theta}(t)\\+w^2(x,t)\ddot{\theta}(t)\\+gw(x,t)\cos(\theta(t))\\-\ddot{\eta}(t)w(x,t)\sin(\theta(t))\\+w_{tt}(x,t)x\end{bmatrix}dx\\&+m_2\begin{bmatrix}2w(l,t)(w_t(l,t))\dot{\theta}(t)\\+w(l,t)^2\ddot{\theta}(t)\\+gw(l,t)\cos(\theta(t))\\-\ddot{\eta}(t)w(l,t)\sin(\theta(t))\\+w_{tt}(l,t)l\end{bmatrix}\\&=\tau(t)+F_3(t)l+d(t)l\end{aligned} \quad (11)$$

$$\begin{aligned}&(\rho l + m_1 + m_2)\ddot{\eta}(t)\\&+\left(\rho\, l^2/2 + m_2 l\right)\ddot{\theta}(t)\cos(\theta(t))\\&-\left(\rho\, l^2/2 + m_2 l\right)\dot{\theta}^2(t)\sin(\theta(t))\\&+\rho\int_0^l\begin{bmatrix}w_{tt}(x,t)\cos(\theta(t))\\-2(w_t(x,t))\dot{\theta}(t)\sin(\theta(t))\\-\ddot{\theta}w(x,t)\sin(\theta)\\-\dot{\theta}^2(t)w(x,t)\cos(\theta)\end{bmatrix}dx\\&+m_2\begin{bmatrix}w_{tt}(l,t)\cos(\theta(t))\\-2w_t(l,t)\dot{\theta}\sin(\theta(t))\\-\ddot{\theta}(t)w(l,t)\sin(\theta(t))\\-\dot{\theta}^2(t)w(l,t)\cos(\theta(t))\end{bmatrix}\\&=F_1(t)\end{aligned} \quad (12)$$

$$\rho\begin{pmatrix}w_{tt}(x,t)+\ddot{\eta}(t)\cos(\theta(t))\\-w(x,t)\dot{\theta}^2(t)\\+g\sin(\theta(t))+\ddot{\theta}(t)x\end{pmatrix}+EIw_{xxxx}=0 \quad (13)$$

$$w(0,t)=w_x(0,t)=w_{xx}(l,t)=0$$

$$m_2\begin{pmatrix} w_{tt}(l,t)+\ddot{\eta}(t)cos(\theta(t)) \\ -w(l,t)\dot{\theta}^2(t)+gsin(\theta(t)) \\ +l\ddot{\theta}(t) \end{pmatrix}-EIw_{xxxx} \quad (14)$$
$$=F_3(t)+d(t)$$

3. Mathematical Preliminaries

For a better understanding of subsequent analysis, some lemmas are mentioned as following:

Lemma 1 (Rahn, 2001). Let $u_1(x,t), u_2(x,t) \in R$ with $x \in [0,L]$ and $t \in [0,\infty]$ the following inequalities hold:

$$2u_1u_2 \le 2|u_1u_2| \le u^2{}_1+u^2{}_2 \;\; \forall u_1,u_2 \in R \qquad (15)$$

Similarly, from (1) we can show that:

$$|u_1u_2| = \left|\left(\frac{1}{\sqrt{\delta}}u_1\right)\left(\sqrt{\delta}u_2\right)\right| \le \frac{1}{\delta}u^2{}_1+\delta u^2{}_2 \qquad (16)$$

Lemma 2 (Rahn, 2001), (Do and Pan, 2008) and (Hardy et al., 1959). Let $u(x,t) \in R$ with $x \in [0,L]$ and $t \in [0,\infty]$ which satisfies the boundary condition $u(0,t)=0$, $t \in [0,\infty]$ $\forall$ the following inequalities hold:

$$\frac{1}{l^2}\int_0^l u^2(x,t)dx \le \int_0^l u^2{}_x(x,t)dx \le l^2\int_0^l u^2{}_{xx}(x,t)dx \qquad (17)$$

$$u^2(x,t) \le l\int_0^l u^2{}_x(x,t)dx \le l^3\int_0^l u^2{}_{xx}(x,t)dx \qquad (18)$$

In addition, if $u(x,t)$ satisfies the boundary condition $u_x(0,t)=0$, the following inequality holds:

$$u_x{}^2(x,t) \le l\int_0^l u^2{}_{xx}(x,t)dx \qquad (19)$$

Equalities (15) , (16), (18) and (19) are utilized through stability analysis to indicate boundedness of Lyapunov function and its time derivative. Based on clamped boundary condition ($w(0,t)=0$), equality (17) is used in order to express that during closed-loop process, all system signals remain bounded.

Assumption 2. Based on kinetic and potential energy in equations (4) and (6) for proposed crane system the following properties hold:

Property 1 (Queiroz et al., 1999, 2001). If the kinetic energy of system is bounded $\forall t \in [0,\infty)$ then $\left(\partial^n/\partial x^n\right)w_t(x,t)$ is bounded for $n=0,1,2$, $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$.

Property 2 (Queiroz et al., 1999, 2001). If the potential energy of system is bounded $\forall t \in [0,\infty)$ then $\left(\partial^n/\partial x^n\right)w_x(x,t)$ is bounded for $n=1,2$, $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$.

Remark 3. From a strictly mathematical point of view, one might question the above boundedness properties. However, from an engineering point of view, it seems reasonable to assume for a real physical system that if the energy of the system is bounded, then all the signals which make up the governing dynamic equations will also remain bounded(de Queiroz et al., 2001)

4. Control design

Based on the control scheme, as shown in Figure 2 the main achievement of this paper is relay on designing boundary signals F_1 , F_2 and $\tau(t)$, which are cooperatively exerted on boundaries and hub. Mentioned signals are employed to analyze the stability of the closed-loop system which leads to achieving the control objectives :(i) drive manipulator's base to the desired position η_d (ii) suppress the transverse vibration of the arm and regulate the rotational angle i.e., to guarantee $w(x,t) \to 0$

and $\theta(t) \to 0$ in the presence of the unknown varying disturbances. The following control laws are proposed:

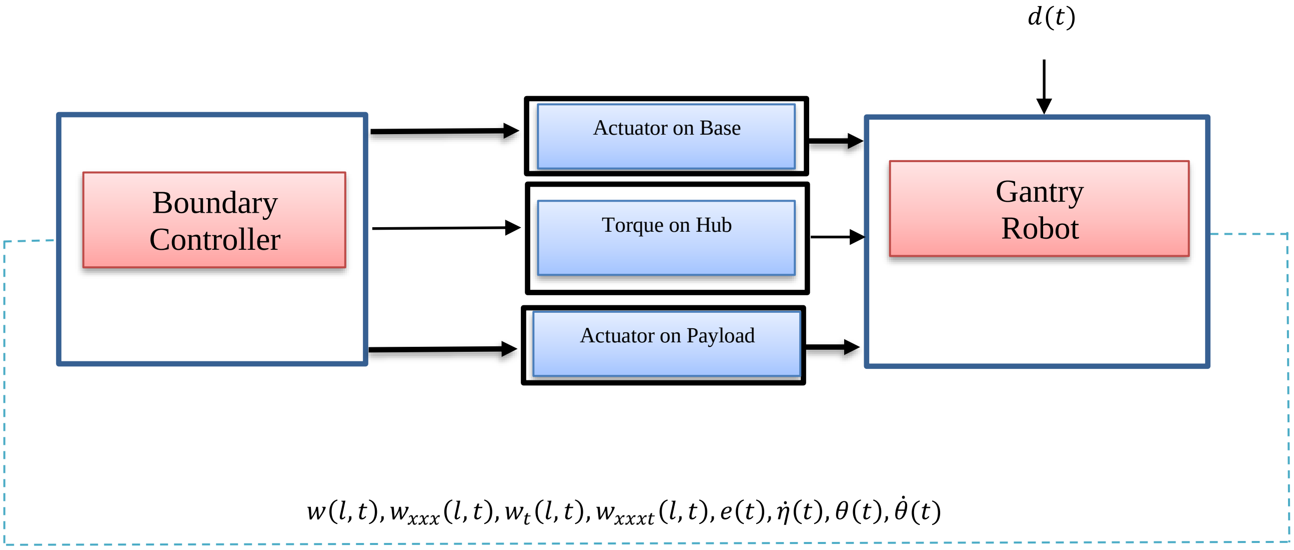

Figure 2 Block diagram of feedback control of flexible gantry manipulator system based on boundary control strategy

$$F_1(t) = -k_1(\varphi(t)) - m_2 w_{xxxt}(l,t) - EIw_{xxx}(l,t) + m_2 g sin(\theta(t)) \quad (20)$$

$$\tau(t) = -k_3 \dot{\theta} w^2(l,t) + (\rho l^2/2) g\, sin(\theta(t)) + m_2\, g\, w(l,t) cos(\theta(t)) \quad (21)$$

$$F_3(t) = -k_2 \dot{\eta}(t) \quad (22)$$

In which k_1, k_2, k_3 and k_4 are positive control gains and auxiliary function $\varphi(t)$ defines as:

$$\varphi(t) = w_t(l,t) + l\dot{\theta}(t) - w_{xxx}(l,t) + k_s \sin^2(\theta(t)) \quad (23)$$

Remark 4. In implementing the control laws, some of the proposed feed backed signals can be measured by existing sensors and the rest can be obtained by calculations(He et al., 2013a). $w_{xxx}(l,t)$ can be measured by shear force sensor at boundary $w_t(l,t)$ and $w_{xxxt}(l,t)$ can be calculated by backward differencing. $\eta(t)$ and $\dot{\eta}(t)$ can be obtained by position and velocity sensors respectively. At last $\theta(t)$ and velocity $\dot{\theta}(t)$ can be determined by rotary encoder and tachometer respectively.

In this work the Lyapunov candidate can be defined as:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + \frac{k_p}{2} e(t)^2 + \frac{k_\theta}{2} sin^2(\theta(t)) \quad (24)$$

Where $\beta, \alpha, k_p, k_\theta$ are positive terms. To achieve control objectives, we define the base position set point error $e(t)$. V_1 is defined based on the total mechanical energy of the string. V_2 is determined based on the kinetic energy of the base and payload mass and at last V_3 is defined in order to competence the stability procedure .they are defined as:

$$V_1 = 1/2\beta\rho\int_0^l \left\{ \begin{pmatrix} w_t(x,t) + x\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right) \end{pmatrix}^2 + \begin{pmatrix} w(x,t)\dot{\theta}(t) \\ -\dot{\eta}(t)sin\left(\theta(t)\right) \end{pmatrix}^2 \right\} dx + \tag{25}$$

$$1/2\beta\int_0^l EIw_{xx}^2(x,t)dx$$

$$V_2 = 1/2\beta m_2 \left\{ \begin{array}{l} \left(\phi(t) + \dot{\eta}(t)cos\left(\theta(t)\right)\right)^2 \\ +\beta\begin{pmatrix} w(l,t)\dot{\theta}(t) \\ -\dot{\eta}(t)sin\left(\theta(t)\right) \end{pmatrix}^2 \end{array} \right\} \tag{26}$$

$$+\beta 1/2 m_1\dot{\eta}^2(t)$$

$$V_3 = \beta\rho g\int_0^l w(x,t)\sin(\theta(t))dx$$

$$+\rho k_s \sin(\theta(t))\int_0^l \begin{pmatrix} w_t(x,t) \\ +x\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right) \end{pmatrix} dx \tag{27}$$

$$+E\,I\,k_s\sin(\theta(t))w(l,t)$$

$$V_4 = \alpha\rho\int_0^l xw_x(x,t)\begin{pmatrix} w_t(x,t) \\ +x\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right) \end{pmatrix} dx \tag{28}$$

Lemma3. The Lyapunov function candidate (24) is bounded by positive variables as following:

$$0 \le \lambda_{11}\begin{pmatrix} \xi(t) + \int_0^l w_{xx}^2dx + \phi^2(t) \\ +e^2(t) + sin^2\left(\theta(t)\right) + \dot{\eta}^2(t) \end{pmatrix} \le V \le \lambda_{12}\begin{pmatrix} \xi(t) + \int_0^l w_{xx}^2dx + \phi^2(t) \\ +e^2(t) + sin^2\left(\theta(t)\right) + \dot{\eta}^2(t) \end{pmatrix} \tag{29}$$

The auxiliary variable $\xi(t)$ is defined as:

$$\xi(t) = \int_0^l w_t^2(x,t)dx + \int_0^l w^2(x,t)\dot{\theta}^2(t)dx + l^2\dot{\theta}^2(t) \tag{30}$$

Proof. Based on equations (16), (25) and adding two positive terms $\frac{k_p}{2}e^2, \frac{k_\theta}{2}sin^2(\theta)$ it is concluded that $V_1 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta)$ is upper bounded as:

$$\begin{aligned} 0 \le V_1 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta) \le \\ 2\beta\rho\int_0^l w_t^2(x,t)dx \\ +1/2\beta\int_0^l EIw_{xx}^2(x,t)dx + 2\beta\rho l\dot{\eta}^2cos^2(\theta) \\ \beta\rho\int_0^l w^2(x,t)\dot{\theta}^2(t)dx + \beta\rho l\dot{\eta}^2sin^2(\theta) \\ +2\beta\rho l^3\dot{\theta}^2 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta) \end{aligned} \tag{31}$$

By having $\begin{array}{l}\beta\rho l\dot{\eta}^2 \le 2\beta\rho l\dot{\eta}^2cos^2(\theta) \\ +\beta\rho l\dot{\eta}^2sin^2(\theta) \le 2l\beta\rho\dot{\eta}^2\end{array}$ and based on definition of $\xi(t)$ in equation (30) we can rewrite equality (31) as:

$$\begin{aligned} \min(\beta\rho, \beta\rho l)\xi(t) + \beta/2\int_0^l EIw_{xx}^2(x,t)dx \\ +\beta\rho l\dot{\eta}^2 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta) \le V_1 \\ \le \max(2\beta\rho, 2\beta\rho l)\xi(t) \\ +\beta/2\int_0^l EIw_{xx}^2(x,t)dx + 2\beta\rho l\dot{\eta}^2 \\ +\frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta) \end{aligned} \tag{32}$$

From definition of V_2 in equation (10) and inequality (1) and by having $b_1 = 2\beta\max\left(\frac{m_{1,}}{2}m_2\right)$ and $b_2 = 2\beta\min\left(\frac{m_{1,}}{2}m_2\right)$ which is positive weighting constant, we can show that is bounded as following:

$$\begin{aligned} b_1\left[\phi^2(t) + \dot{\eta}^2(t) + w^2(l,t)\dot{\theta}^2(t)\right] \le V_2 \\ \le b_2\left[\phi^2(t) + \dot{\eta}^2(t) + w^2(l,t)\dot{\theta}^2(t)\right] \end{aligned} \tag{33}$$

Also in similar manner based on equations (15) , (16) and (27) it is shown that:

$$\begin{aligned}&\left|V_3+V_4\right|\le 2\alpha\rho l\,\xi(t)\\&+\alpha\rho l\int_0^l w_{xx}{}^2(x,t)\,dx\\&+2\alpha\rho l\,\dot\eta^2 cos^2(\theta)\\&+\beta\rho g\,\sin^2(\theta(t))\\&+\beta\rho g\int_0^l w^2(x,t)\,dx\end{aligned}\tag{34}$$

Based on relations (17) and (18) it can be shown $\int_0^l w^2(x,t)dx \le l^4\int_0^l w^2{}_{xx}(x,t)dx$ and $w^2(l,t)\le l^3\int_0^l w^2{}_{xx}(x,t)dx$ respectively. So relation (34) can be rewritten as:

$$\begin{aligned}&\left|V_3+V_4\right|\le\\&4+\left(k_s\left(\rho+EIl^3\right)+\beta\rho g l^2+\alpha\rho l\right)\\&\times\int_0^l w^2{}_{xx}(x,t)dx\,\left(\alpha\rho l+\rho k_s\right)\xi(t)\\&+4\rho l k_s\dot\eta^2+\\&\left(\rho k_s+\beta\rho g+EI\right)\sin^2(\theta(t))\end{aligned}\tag{35}$$

By adding equalities (31) ,(33) and (35) based on defining positive constants as following:

$\lambda_1=\min(\beta\rho,\beta\rho l)-4(\alpha\rho l+\rho k_s)>0$

$\lambda_2=2\max(\beta\rho,\beta\rho l)+4(\alpha\rho l+\rho k_s)$

$\lambda_3=\beta/2\,EI-\left(k_s\left(\rho+EIl^3\right)+\beta\rho g l^2+\alpha\rho l\right)>0$

$\lambda_4=\beta/2\,EI+\left(k_s\left(\rho+EIl^3\right)+\beta\rho g l^2+\alpha\rho l\right)$

Furthermore

$\lambda_5=b_1+2\beta\rho l-4l\left(\alpha\rho+\rho k_s\right)>0$ and $\lambda_6=b_1+\beta\rho+4l\left(\alpha\rho+\rho k_s\right)$. Considering, $\lambda_7=\dfrac{k_\theta}{2}-\left(\rho k_s+\beta\rho g+EI\right)>0$ and $\lambda_8=\dfrac{k_\theta}{2}+\left(EI+\rho k_s+\beta\rho g\right)$, the Lyapunov candidate function relation (29) is proved. where $\lambda_9=\min(\lambda_1,\lambda_3,\lambda_5,\lambda_7)$ and $\lambda_{10}=\max(\lambda_2,\lambda_4,\lambda_6,\lambda_8)$ are positive constants.

Theorem 1. For the system dynamics described by (11) , (12) and (13) , boundary conditions (14), using the proposed boundary control laws (20), (21) and (22) , if the initial conditions are bounded, then the system is regulated asymptotically in the following sense:

$$\lim_{t\to\infty}\left|w(x,t)\right|=0\,,\quad \lim_{t\to\infty}\left|\theta(t)\right|=0\,,\quad \lim_{t\to\infty}\left|e(t)\right|=0$$

Proof. Differentiating Lyapunov candidate equation (24) respect to time leads to:

$$\begin{aligned}&\dot V(t)=\dot V_1(t)+\dot V_2(t)+\dot V_3(t)\\&+\dot V_4(t)+k_\theta\dot\theta(t)sin(\theta)\end{aligned}\tag{36}$$

The first term of equation (36) can be represented as:

$$\dot V_1(t)=E_1(t)+E_2(t)\tag{37}$$

In which $E_1(t),E_2(t)$ can be shown as:

$$\begin{aligned}E_1=\beta\rho\int_0^l&\left\{\begin{matrix}(w_t)(w_{tt})+(w_t)w\,\dot\theta^2\\+\ddot\eta(w_t)cos(\theta)+(w_t)\ddot\theta x\end{matrix}\right\}dx\\&+\beta\int_0^l EIw_{xxt}w_{xx}dx\end{aligned}\tag{38}$$

$$\begin{aligned}E_2=\beta\rho\int_0^l&\{\dot\eta\ddot\eta+x\,\dot\theta\ddot\theta+w^2\dot\theta\ddot\theta\\&+\ddot\eta\dot\theta xcos(\theta)+\dot\eta\ddot\theta xcos(\theta)\\&-\dot\eta\dot\theta^2 xsin(\theta)+\\&\dot\eta(w_{tt})cos(\theta)-\dot\eta(w_t)\dot\theta sin(\theta)\\&-\ddot\eta\dot\theta wsin(\theta)-\dot\eta\ddot\theta wsin(\theta)\\&-\dot\eta\dot\theta(w_t)sin(\theta)\\&-\dot\eta\dot\theta^2 wcos(\theta)+(w_{tt})\dot\theta x\}dx\end{aligned}\tag{39}$$

By substituting equation of motion (13) into equation (38) we obtain:

$$\begin{aligned}E_1(t) &= -\beta\int_0^l EIw_t(x,t)w_{xxxx}(x,t)dx \\ &+\beta\int_0^l EI\, w_{xxt}w_{xx}dx \\ &+2\beta\rho\int_0^l w_t\, w\,\dot{\theta}^2 dx \\ &-\beta\rho\int_0^l g\left(w_t\right)sin\left(\theta(t)\right)dx\end{aligned} \tag{40}$$

By integration by part and from the definition of auxiliary function $\varphi(t)$, equation (40) yields to:

$$\begin{aligned}E_1 &\le \beta EI/2\varphi(t)^2 \\ &-\beta EI\begin{pmatrix} w^2_{xxx}(l,t)+w^2_t(l,t) \\ +l^2\dot{\theta}^2+k_s^{\,2}\sin^2(\theta(t)) \end{pmatrix} \\ &-\beta EI\begin{pmatrix} w_t(l,t)+k_s\sin(\theta(t)) \\ +w_{xxx}(l,t) \end{pmatrix} l\dot{\theta} \\ &-EIk_s\sin(\theta(t))\begin{pmatrix} -w_{xxx}(l,t) \\ +w_t(l,t) \end{pmatrix} \\ &-\beta\rho\int_0^l g\left(w_t\right)sin\left(\theta(t)\right)dx\end{aligned} \tag{41}$$

Based on (16), differentiating $V_2(t)$ yields to:

$$\begin{aligned}\dot{V}_2 &= \beta m_2\left[\phi(t)\begin{pmatrix} w_{tt}(l,t) \\ -\dot{\eta}(t)\dot{\theta}(t)sin\left(\theta(t)\right) \\ +l\,\ddot{\theta}(t) \\ +\ddot{\eta}(t)cos\left(\theta(t)\right)\end{pmatrix}\right] \\ &+\beta m_2\begin{bmatrix} w_{tt}(l,t)\begin{pmatrix} l\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right)\end{pmatrix} \\ +l\,\ddot{\theta}(t)\dot{\eta}(t)cos\left(\theta(t)\right)\end{bmatrix} \\ &+\beta m_2\left[l\,\dot{\theta}(t)\begin{pmatrix} l\,\ddot{\theta}(t)+\ddot{\eta}cos\left(\theta(t)\right) \\ -l\,\dot{\theta}(t)\dot{\eta}(t)sin\left(\theta(t)\right)\end{pmatrix}\right] \\ &+\beta m_2\begin{bmatrix} w\left(w_t\right)\dot{\theta}^2 \\ +w^{\,2}\ddot{\theta}\dot{\theta}-\ddot{\eta}w\,\dot{\theta}sin\left(\theta\right) \\ -\dot{\eta}w_t\dot{\theta}sin\left(\theta\right) \\ -\dot{\eta}w\,\ddot{\theta}sin\left(\theta\right)-\dot{\eta}w\,\dot{\theta}^2cos\left(\theta\right)\end{bmatrix} \\ &+\beta(m_1+m_2)\dot{\eta}(t)\ddot{\eta}(t) \\ &+\beta m_2 w_{xxxt}(l,t)\phi^2(t)\end{aligned} \tag{42}$$

Differentiating $V_3(t)$ yields to:

$$\begin{aligned}\dot{V}_3(t) &= \left(k_s\,\dot{\theta}cos\left(\theta\right)\right)\int_0^l\begin{pmatrix} w_t(x,t)+x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right)\end{pmatrix}dx \\ &+\left(k_s\sin(\theta)\right)\int_0^l\begin{pmatrix} w_{tt}(x,t)+x\,\ddot{\theta}(t) \\ +\ddot{\eta}(t)cos\left(\theta(t)\right) \\ -\dot{\eta}(t)\,\dot{\theta}(t)sin\left(\theta(t)\right)\end{pmatrix}dx \\ &+\beta\,\rho\,g\int_0^l w_t(x,t)\sin(\theta(t))dx \\ &+\beta\,\rho\,g\int_0^l w(x,t)\,\dot{\theta}(t)\cos(\theta(t))dx \\ &-E\,I\,k_s\dot{\theta}cos\left(\theta\right)w(l,t) \\ &+E\,I\,k_s\sin(\theta(t))w_t(l,t)\end{aligned} \tag{43}$$

By substituting equation of motion (13) into equation (43) we obtain:

$$\dot{V}_3(t) = k_s \sin(\theta)\int_0^l \begin{bmatrix} -EIw_{xxxx}(x,t) \\ +\rho w\,\dot{\theta}^2 \\ -\rho g sin(\theta) \\ -\rho\dot{\eta}\dot{\theta} sin(\theta) \end{bmatrix} dx$$
$$+k_s\dot{\theta}cos(\theta)\int_0^l \begin{pmatrix} w_t(x,t)+x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx$$
$$+\beta\rho g\int_0^l w_t(x,t)\sin(\theta(t))dx \qquad (44)$$
$$+\beta\rho g\,\dot{\theta}(t)cos(\theta(t))\int_0^l w(x,t)dx$$
$$+\beta\rho g\int_0^l w_t(x,t)\sin(\theta(t))dx$$
$$+\beta\rho g\int_0^l w(x,t)\dot{\theta}(t)\cos(\theta(t))dx$$
$$-EIk_s\dot{\theta}(t)\cos(\theta(t))w(l,t)$$
$$+\beta EIk_s\sin(\theta)w_t(l,t)$$

Furthermore based on $0 \le cos(\theta) \le 1$ we have:

$$k_s\sin(\theta)\int_0^l \begin{bmatrix} -EIw_{xxxx}(x,t) \\ +\rho w\,\dot{\theta}^2 \\ -\rho g sin(\theta) \\ -\rho\dot{\eta}\dot{\theta} sin(\theta) \end{bmatrix} dx \le$$
$$\frac{\rho k_s}{\delta_1}\dot{\theta}^2 + k_s\rho\delta_1\int_0^l w^2\dot{\theta}^2 dx \qquad (45)$$
$$-k_s\rho gl\sin^2(\theta)$$
$$-k_s\rho l\,\dot{\eta}\dot{\theta} sin^2(\theta)$$
$$-k_s EIw_{xxx}(l)\sin(\theta)$$

$$k_s\rho\dot{\theta}cos(\theta)\int_0^l \begin{pmatrix} w_t(x,t)+x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx \le$$
$$k_s\rho\delta_2\dot{\theta}^2 + \frac{\rho k_s}{\delta_2}\int_0^l w_t^2(x,t)dx \qquad (46)$$
$$+k_s\rho\, l^2/2\,\dot{\theta}^2 + k_s\rho\, l\,\dot{\eta}(t)\dot{\theta}cos^2(\theta)$$

$$\beta\rho g\,\dot{\theta}(t)cos(\theta(t))\int_0^l w(x,t)dx \le$$
$$\beta\rho g\,\delta_3\dot{\theta}^2(t) + \frac{\beta\rho g l^4}{\delta_3}\int_0^l w^2_{xx}(x,t)dx \qquad (47)$$

In similar way by differentiating $V_4(t)$ one can attain:

$$\dot{V}_4 = \alpha\rho\int_0^l x w_{xt}(x,t) \begin{pmatrix} w_t(x,t)+ \\ x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx$$
$$+\alpha\rho\int_0^l x w_x(x,t) \begin{pmatrix} w_{tt}(x,t) \\ +x\,\ddot{\theta}(t) \\ +\ddot{\eta}(t)cos(\theta(t)) \\ -\dot{\eta}(t)\dot{\theta}(t)sin(\theta(t)) \end{pmatrix} dx \qquad (48)$$

By substituting equation of motion (13) into (48) and some simplification yields to:

$$\dot{V}_4 = \alpha\int_0^l \rho x w_{xt}(x,t) \begin{pmatrix} w_t(x,t) \\ +x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx$$
$$-\alpha EI\int_0^l x w_x(x,t)w_{xxxx}(x,t)dx \qquad (49)$$
$$+\alpha\int_0^l \rho x w_x \begin{pmatrix} w\,\dot{\theta}^2 + g sin(\theta) \\ -\dot{\eta}\dot{\theta} sin(\theta) \end{pmatrix} dx$$

Equation (49) can represented as:

$$\dot{V}_4(t) = A_1(t) + A_2(t) + A_3(t) \qquad (50)$$

which:

$$A_1(t) = \alpha\int_0^l \rho x w_{xt}(x,t) \begin{pmatrix} w_t(x,t) \\ +x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx \qquad (51)$$

$$A_2(t) = \alpha\int_0^l \rho x w_x \begin{pmatrix} w\,\dot{\theta}^2 \\ +g sin(\theta) \\ -\dot{\eta}\dot{\theta} sin(\theta) \end{pmatrix} dx \qquad (52)$$

$$A_3(t) = -\alpha EI\int_0^l \{x\, w_x(x,t)\, w_{xxxx}(x,t)\}dx \qquad (53)$$

Integrating equation (51) leads to:

$$A_1(t)=\alpha\rho l\,\frac{w_t^{\,2}(l,t)}{2}-\alpha\rho l\int_0^l\frac{w_t^{\,2}}{2}dx$$
$$+\alpha\rho l^2\dot{\theta}w_t-2\alpha\rho l\int_0^l x\,\dot{\theta}w_t dx \tag{54}$$
$$+\alpha\int_0^l\rho x w_{xt}\left(\dot{r}cos(\theta)\right)dx$$

Based on relation (16) it can be shown that:

$$\alpha\int_0^l\rho x w_{xt}(x,t)\left(\dot{\eta}(t)cos(\theta(t))\right)dx<$$
$$\frac{\alpha\rho l}{\sigma_1}w_t^{\,2}(l,t)$$
$$+\alpha\rho\sigma_2\sqrt{\frac{l^3}{3}}\int_0^l w_t(x,t)^2dx \tag{55}$$
$$\left(\alpha\rho l\sigma_1+\frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2(\theta(t))$$

$$\alpha\rho l^2\dot{\theta}w_t(l,t)$$
$$+2\alpha\rho\int_0^l x\,\dot{\theta}w_t(x,t)dx\le$$
$$\left(\frac{\alpha\rho l^2}{\sigma_3}+2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\dot{\theta}^2(t) \tag{56}$$
$$\times\left(2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\int_0^l w_t^{\,2}(x,t)dx$$
$$+\alpha\rho l^2\sigma_3 w_t^{\,2}(l,t)$$

By substituting relations (55) and (56) into (54) it can be obtained:

$$A_1(t)\le-\left(\frac{\alpha\rho l}{2}-2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4}-\alpha\rho\sigma_2\sqrt{\frac{l^3}{3}}\right)$$
$$\int_0^l w_t^{\,2}(x,t)dx$$
$$+\left(\frac{\alpha\rho l}{2}+\frac{\alpha\rho l}{\sigma_1}+\alpha\rho l^2\sigma_3\right)w_t^{\,2}(l,t) \tag{57}$$
$$\left(\frac{\alpha\rho l^2}{\sigma_3}+2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\dot{\theta}^2(t)$$
$$+\left(\alpha\rho l\sigma_1+\frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2(\theta(t))$$

Integrating equation (52) leads to:

$$A_2(t)=\frac{\alpha\rho l}{2}\dot{\theta}^2w^2(l,t)$$
$$-\alpha\rho\dot{\theta}^2\int_0^l w^2dx \tag{58}$$
$$+\alpha\int_0^l\rho x\,w_x\left(gsin(\theta)-\dot{\eta}\dot{\theta}sin(\theta)\right)dx$$

By integrating the third part of equation (58) on can obtain:

$$\alpha\int_0^l\rho x w_x\left(gsin(\theta)-\dot{\eta}\dot{\theta}sin(\theta)\right)dx\le$$
$$-\alpha\rho l\,\dot{\eta}(t)\dot{\theta}(t)w(l,t)sin(\theta)(t)$$
$$+\alpha\rho l\,\dot{\eta}(t)\dot{\theta}(t)sin(\theta(t))\int_0^l w(x,t)dx \tag{59}$$
$$\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}sin^2(\theta)$$
$$+\alpha\rho g\,\sigma_5\sqrt{\frac{l^3}{3}}\int_0^l w_x(x,t)dx$$

Relation (59) also can be rewritten as:

$$\alpha\int_0^l\rho x w_x\left(gsin(\theta)-\dot{\eta}\dot{\theta}sin(\theta)\right)dx\le$$
$$\left(\alpha\rho l\sigma_6+\frac{\alpha\rho l}{\sigma_7}\right)\dot{\eta}^2(t)sin^2(\theta(t))$$
$$+\alpha\rho l\sigma_7\dot{\theta}^2(t)\int_0^l w^2(x,t)dx \tag{60}$$
$$+\frac{\alpha\rho l}{\sigma_6}\dot{\theta}^2(t)w^2(l,t)\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}sin^2(\theta)$$
$$+\alpha\rho g\,\sigma_5 l^2\sqrt{\frac{l^3}{3}}\int_0^l w_{xx}(x,t)dx$$

Based on inequality (60), equation (58) yields to:

$$A_2(t) \le -\left(\alpha\rho - \alpha\rho l\,\sigma_7\right)\dot{\theta}^2\int_0^l w^2 dx$$
$$+\left(\alpha\rho\frac{l}{2}+\frac{\alpha\rho l}{\sigma_6}\right)\dot{\theta}^2(t)w^2(l,t)$$
$$\left(\alpha\rho l\,\sigma_6+\frac{\alpha\rho l}{\sigma_7}\right)\dot{\eta}^2(t)sin^2(\theta(t)) \quad (61)$$
$$+\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}sin^2(\theta)$$
$$+\alpha\rho g\,\sigma_5 l^2\sqrt{\frac{l^3}{3}}\int_0^l w_{xx}(x,t)dx$$

By integrating by parts, $A_3(t)$ yields to:

$$A_3(t) = -\alpha EILw_x(l,t)w_{xxx}(l,t)$$
$$+\alpha EI\int_0^L w_x(x,t)w_{xxx}(x,t)dx \quad (62)$$
$$+\alpha EI\int_0^L x\, w_{xx}(x,t)w_{xxx}(x,t)dx$$

By integrating relation(62), $A_2(t)$ can be obtained as follows:

$$A_3 \le -\alpha EILw_x(l,t)w_{xxx}(l,t)$$
$$-3\alpha EI\ /\ 2\int_0^L w^2_{xx}(x,t)\,dx \quad (63)$$

Based on (16) and (19) Relation (63) also can be rewritten as:

$$A_3 \le \frac{\alpha EIl}{\sigma_8}w^2_{xxx}(l,t)$$
$$-\alpha EI\left(3/2 - l\,\sigma_8\right)\int_0^L w^2_{xx}(x,t)dx \quad (64)$$

By substituting (57), (61) and (64) into (50), $\dot{V}_4$ can be determined as following:

$$\dot{V}_4 \le -\alpha\left(EI\left(3/2 - l\,\sigma_8\right) - \rho g\,\sigma_5 l^2\sqrt{\frac{l^3}{3}}\right)$$
$$\int_0^l w_{xx}^2(x,t)dx$$
$$-\left(\alpha\rho - \alpha\rho l\,\sigma_7\right)\dot{\theta}^2\int_0^l w^2 dx$$
$$-\left(\frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}}\right)$$
$$\int_0^l w_t^2(x,t)dx$$
$$+\left(\frac{\alpha\rho\, l}{2}+\frac{\alpha\rho l}{\sigma_1}+\alpha\rho l^2\sigma_3\right)$$
$$w_t^2(l,t)\frac{\alpha EIl}{\sigma_8}w_{xxx}^2(l,t) \quad (65)$$
$$+\left(\frac{\alpha\rho l^2}{\sigma_3}+2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\dot{\theta}^2(t)$$
$$+\left(\alpha\rho\frac{l}{2}+\frac{\alpha\rho l}{\sigma_6}\right)\dot{\theta}^2(t)w^2(l,t)$$
$$+\left(\alpha\rho l\,\sigma_1+\frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2(\theta(t))$$
$$+\left(\alpha\rho l\,\sigma_6+\frac{\alpha\rho l}{\sigma_7}\right)\dot{\eta}^2(t)sin^2(\theta(t))$$
$$+\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}sin^2(\theta)$$

By introducing following relations as follows:

$$\left(\alpha\rho l\,\sigma_1+\frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2(\theta(t))$$
$$+\left(\alpha\rho l\,\sigma_6+\frac{\alpha\rho l^2}{\sigma_7}\right)\dot{\eta}^2(t)sin^2(\theta(t)) \quad (66)$$
$$\le \lambda\left(\dot{\eta}^2 sin^2(\theta)+\dot{\eta}^2 cos^2(\theta)\right) \le \lambda\dot{\eta}^2$$

In which

$$\lambda = \max\left(\alpha\rho l\,\sigma_1+\frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}},\alpha\rho l\,\sigma_6+\frac{\alpha\rho l^2}{\sigma_7}\right)$$

and also:

By substituting equations (64), (65) and (66) into (36), then we have:

$$
\begin{aligned}
\dot{V} \le\; & -\begin{pmatrix} \alpha EI\left(3/2 - l\sigma_8\right) \\ -\alpha\rho g\sigma_5 l^2\sqrt{\frac{l^3}{3}} - \frac{\beta\rho g l^4}{\delta_3} \end{pmatrix}\int_0^l w_{xx}^{\,2}(x,t)dx \\
& -\left(\alpha\rho - \alpha\rho l\sigma_7 - k_s\rho\delta_1\right)\dot{\theta}^2(t) \\
& \times\int_0^l w^2(x,t)dx \\
& -\left(\frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}} - \frac{\rho k_s}{\delta_2}\right) \\
& \times\int_0^l w_t^{\,2}(x,t)dx \\
& -\left(\beta k_1 - \beta EI/2\right)\varphi(t)^2 \\
& -\begin{pmatrix} \beta k_2 + \frac{\beta EIl^2}{2} - \alpha\rho\left(\frac{l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\sigma_4\right) \\ + \frac{\rho k_s}{\delta_1} - k_s\rho\delta_2 - k_s\rho\, l^2/2 - \beta\rho g\,\delta_3 \end{pmatrix}\dot{\theta}^2(t) \\
& -\left(\beta k_3 - \alpha\rho\frac{l}{2} - \frac{\alpha\rho l}{\sigma_6}\right)\dot{\theta}^2(t) w^2(l,t) \\
& -\left(\frac{\beta EI}{2} - \frac{\alpha EIl}{\sigma_8}\right)w_{xxx}^{\,2}(l,t) - \left(\beta k_4 - \lambda\right)\dot{\eta}^2 \\
& -\left(\frac{\beta EI}{2} - \frac{\alpha\rho l}{2} - \frac{\alpha\rho l}{\sigma_1} - \alpha\rho l^2\sigma_3\right)w_t^{\,2}(l,t) \\
& -\left(k_s^{\,2} + k_s\rho g l - \frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}\right)\sin^2(\theta)
\end{aligned} \tag{67}
$$

The positive parameters $k_1 - k_4$, k_p, k_θ, k_{e1}, k_s, $\delta_{1-11}, \sigma_{1-8}$, α, and β are determined in order to fulfill following relations:

$$
\upsilon_1 = \alpha EI\left(3/2 - l\sigma_8\right) - \alpha\rho g\sigma_5 l^2\sqrt{\frac{l^3}{3}} - \frac{\beta\rho g l^4}{\delta_3} > 0 \tag{68}
$$

$$
\upsilon_2 = \alpha\rho - \alpha\rho l\sigma_7 - k_s\rho\delta_1 > 0 \tag{69}
$$

$$
\upsilon_3 = \frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}} - \frac{\rho k_s}{\delta_2} > 0 \tag{70}
$$

$$
\upsilon_4 = \frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}} - \frac{\rho k_{e1}}{\delta_4} - \frac{\rho k_s}{\delta_9} > 0 \tag{71}
$$

$$
\upsilon_5 = \beta k_1 - \beta EI/2 > 0 \tag{72}
$$

$$
\upsilon_6 = \beta k_2 + \frac{\beta EIl^2}{2} - \alpha\rho\left(\frac{l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\sigma_4\right) + \frac{\rho k_s}{\delta_1} - k_s\rho\delta_2 - k_s\rho\, l^2/2 - \beta\rho g\,\delta_3 > 0 \tag{73}
$$

$$
\upsilon_7 = \beta k_3 - \alpha\rho\frac{l}{2} - \frac{\alpha\rho l}{\sigma_6} > 0 \tag{74}
$$

$$
\upsilon_8 = \frac{\beta EI}{2} - \frac{\alpha EIl}{\sigma_8} > 0 \tag{75}
$$

$$
\upsilon_9 = \beta k_4 - \lambda > 0 \tag{76}
$$

$$
\upsilon_{10} = \frac{\beta EI}{2} - \frac{\alpha\rho l}{2} - \frac{\alpha\rho l}{\sigma_1} - \alpha\rho l^2\sigma > 0 \tag{77}
$$

$$
\upsilon_{11} = \frac{\beta EI}{2} - \frac{\alpha\rho l}{2} - \frac{\alpha\rho l}{\sigma_1} - \alpha\rho l^2\sigma_3 > 0 \tag{78}
$$

$$
\upsilon_{12} = k_s^{\,2} + k_s\rho g l - \frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}} > 0 \tag{79}
$$

Also relation (67) implies that $\dot{V} \le -\lambda_{11}\begin{pmatrix}\xi(t) \\ +\int_0^l w^2{}_{xx}(x,t)dx \\ +\phi^2(t)+\dot{\eta}(t)^2+\sin^2(\theta)\end{pmatrix}$, in which λ_{13} is positive constant by definition $\lambda_{13}=\min(\upsilon_1,\upsilon_2,\upsilon_3,...,\upsilon_{12})$. Also there exist

$$\xi(t)+\int_0^l w^2{}_{xx}(x,t)dx+\phi^2(t)+e^2+\dot{\eta}(t)^2+\sin^2(\theta)$$

is bounded. Because all terms are all positive, then $\xi(t), w^2{}_{xx}(x,t), \phi^2(t)$ and $\dot{\eta}(t)^2$ are all bounded, based on boundedness of $\xi(t)$, $w_t(x,t)$, $\dot{\theta}(t)$ are bounded $\forall t\in[0,\infty)$ and $\forall x\in[0,l]$. So boundedness of the total mechanical energy of the string system is obtained. From properties 1 and 2, $w_{xxxt}(x,t)$ and $w_{xxxx}(x,t)$ are bounded $\forall t\in[0,\infty)$ and $\forall x\in[0,l]$. From equation of motions (11), (12) and (13) it will be concluded that $w_{tt}(x,t)$, $\ddot{\eta}(t)$ and $\ddot{\theta}(t)$ are all bounded $\forall t\in[0,\infty)$ and $\forall x\in[0,l]$. Based on all above statements it can be concluded that the controls signals are bounded and all the signals in closed loop system remains bounded.

5. Simulation Results

Based on finite difference method, the solution for proposed system which has been described by (11), (12) and (13) with the boundary conditions (14) is approximated. This method provides an accurate process for solving differential equations (He, He, et al., 2015). Consider proposed system excited by boundary disturbance $d(t)=0.1\sin(0.1\pi t)$ with initial conditions $\theta_0=40^0$, $\eta_0=0$, $w(x,0)=0$ and $w_t(x,0)=0$. The objective of this section is to demonstrate the performance of the proposed control laws (20), (21), (22) and disturbance observer. The other specifications are presented in Table 1.

Table 1. Gantry manipulator specifications

Parameter	Description	Value
l	Length of manipulator	$0.8\ m$
m_1	Mass of base	$5\ kg$
m_2	Mass of tip payload	$4\ kg$
ρ	Uniform mass per unit length	$0.2\ kg/m$
EI	Bending stiffness of manipulator	$8\ Nm^2$
I	Inertial of the hub	$0.3\ kg/m^2$
η_d	Desired position	$1\ m$

In order to evaluate the control performance, two cases are compared. First, the behavior of the system without control inputs is shown. Then the effectiveness of proposed control inputs is illustrated by applying them to the system. The manipulator position, the threedimensional representation of the transverse vibration of the manipulator, the deflection of the four equi-distanced points on the link, and the angular position are shown in Figures 4-7 respectively. This implicates that the system is thoroughly unstable in the absence of feedback control inputs.

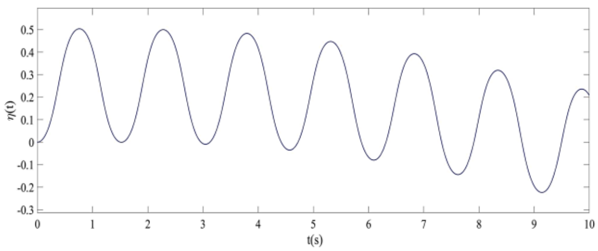

Figure 3. Manipulator position without control

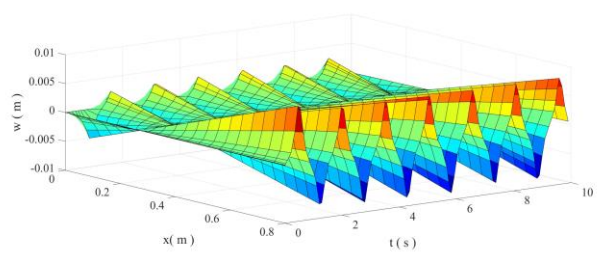

Figure 5. Transverse displacement of flexible manipulator without control

As it is shown, due to boundary disturbance, the manipulator position is far beyond the desired position and the angular position is far away from regulation point. Moreover, the flexible manipulator is subjected to the large vibrations relative to its length which cannot be ignored.

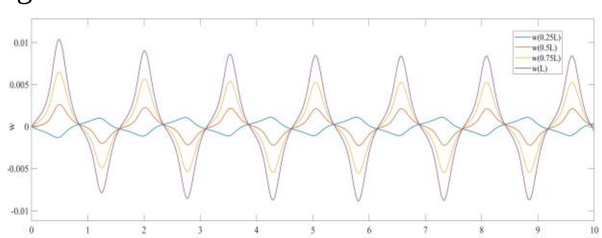

Figure 4. The deflection of the four equi-distanced points on the link without control

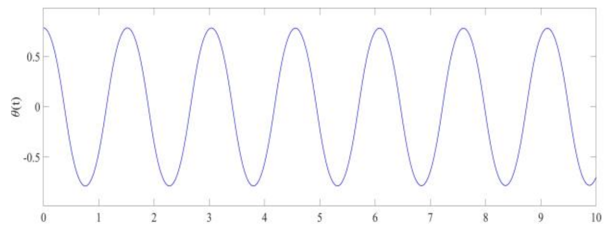

Figure 7. The angular position of flexible manipulator without control

In the second case, in order to reach the control objectives and guarantee the best performance of the controller, the simulation performed by applying proposed control laws with proper parameters. Mentioned control gains are taken cautiously by a large number of trials. Furthermore, they must satisfy relations $\delta_{1-11}, \sigma_{1-8}$ and (29) which all lead to reach the best performance of the controller.

Accordingly, the proposed parameters are chosen as $\beta = 3$, $\alpha = 0.01$, $k_{e1} = 0.02$, $k_s = 0.01$, $\delta_{1-2} = 10$, $\delta_3 = 1$, $\delta_4 = 0.52$, $\delta_5 = 98$, $\delta_{6-9} = 1$, $\delta_{10} = 0.1$, $\delta_{11} = 1$, $\delta_{12} = 26$, $\delta_{13} = 12.4$, $\delta_{14-15} = 1$, $k_1 = 1100$, $k_2 = 7$, $k_3 = 1.5$, $k_4 = 1$, $k_\theta = 70$ and $k_p = 260$.

Figure 8 shows that the system is driven to the desired position within 6 seconds. It is noticeable, the deflections due to the vibrations, are suppressed as a result of the feedback proposed control inputs shown in the figures 9-10. Moreover, according to Figure 11 the angular position is greatly regulated within 7 Sec.

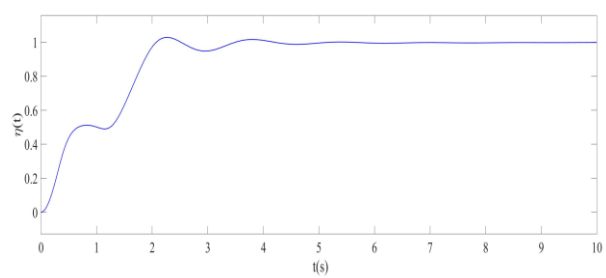

Figure 5. Manipulator position with control

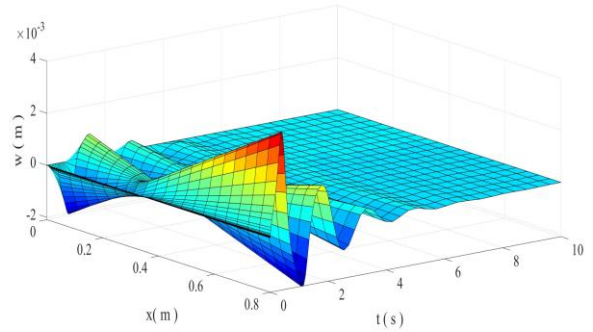

Figure 6. Transverse displacement $w(x,t)$ of flexible manipulator without control

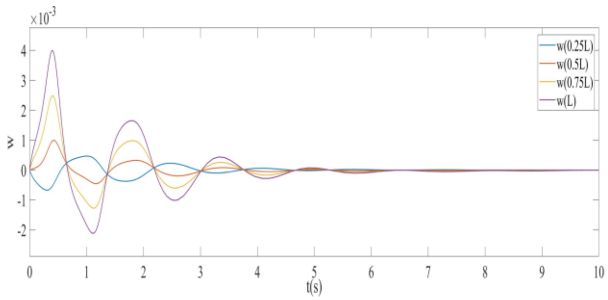

Figure 7. The deflection of the four equi-distanced points on the link without control

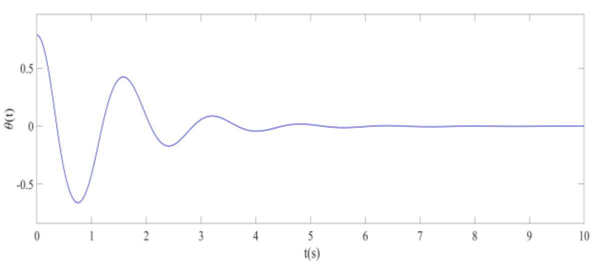

Figure 8. The angular position of flexible manipulator without control

6. Conclusion

In this paper, based on Hamiltonian, dynamic PDE-ODE representation of a flexible gantry robot manipulator under time-varying boundary disturbance with rigid body nonlinear large angular position and translation has been derived. In order to achieve precise locating of system, a new boundary control procedure has been introduced based on the original hybrid PDE-ODE dynamic modelling without any simplification in which all nonlinearities and gravitational force has been considered. Indeed, based on a novel proposed control schemes, by designing boundary control laws and boundary disturbance observer, transverse vibration and nonlinear angular position have been regulated with exponential decay rate, manipulator steered to the desired position, and boundary disturbance has estimated simultaneously. Based on the Lyapunov direct method, the ultimate boundedness of closed-loop system has been achieved. Numerical simulations illustrate the effectiveness of the nonlinear proposed controller and observer.

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