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A new approach for solving the direct kinematic problem of a general 3-RRR spherical parallel robot

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ABSTRACT

Various structures for the spherical parallel robots have been proposed. The 3-RRR Spherical parallel robot and its specific structures like Agile Eye/Wrist is one of the most famous spherical parallel robots. In this article, a new approach is proposed for modeling the direct kinematic problem of this robot to obtain all assembly modes. Utilizing the spherical geometry of the robot, two coupled trigonometric equations are obtained using the angle-axis representation. Next, the two coupled equations are solved using Sylvester's elimination method which leads to a polynomial of eight degrees. Finally, two examples are provided which have eight real solutions (assembly modes) and confirming the assembly modes is performed by a commercial modeling software package. The eight real solutions can be concluded that the degree of the obtained polynomial is the minimum and the proposed modeling is optimal. The presented algorithm is very valuable for direct kinematic analysis of the spherical parallel manipulators due to its straightforward implementation and simplicity. The step-by-step and conceptual approach of the presented method is also applicable to various parallel structures.

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1. Introduction

Many industrial applications require orientating a rigid body around a fixed point such as; orienting a tool or a workpiece in machine tools, solar panels, space antenna and telescopic mechanisms, flight simulator mechanism and camera devices. A spherical manipulator is one in which the end-effector is moved on the surface of a sphere. In other words, the end-effector can rotate around any axis passing through a fixed point, the center of the sphere. Therefore a spherical manipulator can be used as a device to orient the end-effector[1-11].

Applications of such robots can include mechanisms for determining the direction of machine tools or workpieces, solar panels, space aerials and telescopic mechanisms, simulation of three degrees of freedom systems, functional test of an autopilot of a rocket, rehabilitation robot for the wrist, ankle and shoulder. Generally, rigid body orientation without any changes in its position is required in many technical applications.

Various structures for spherical parallel robots have been proposed. Alici and B. Shirinzadeh [13] proposed a spherical parallel robot with structure 3SPS-S. The branches of this robot are the same and each branch is composed of SPS joints. In each branch, a spherical joint is attached to the fixed base and another spherical joint is connected to the moving platform. These spherical joints are non-actuating. The moving plate is also connected directly to the fixed plate by a non-actuating spherical joint. In this robot, prismatic joints of each branch are considered as actuator joints.

C. Innocenti and V. Parenti-Castelli [13], J. Enferadi[14], K. Wohlhart[15] and R. Vertechy and V. Parenti-Castelli [16] studied the spherical parallel robot with structure 3UPS–S. The branches of this robot are the same and each branch is composed of UPS joints. In each branch, the universal joint is attached to the fixed base and the spherical joint is connected to the moving platform. These joints are non-actuating. The moving platform is directly connected to the fixed platform through a passive spherical joint. Also, the prismatic joints in each kinematic chain are regarded as active joints.

R. Di Gregorio [17] proposed a new spherical parallel robot with a 3-RRS structure. The robot consists of identical branches, with each branch comprising RRS joints. In each branch, a revolute joint is attached to the fixed base and a spherical joint is connected to the moving platform. All the revolute joints of this robot are on a hypothetical sphere. The rotational axes of all the revolute joints pass from the center of the sphere. In this robot, the revolute joints that are connected to the base are considered as actuator joints. M. Karouia and J.M. Herve [18] proposed an asymmetrical parallel robot. The connection arrangement of each branch of this robot is different from its other branches. In the first branch, there are three revolute joints in which their axes pass from one point. In the second branch, there are three prismatic joints and a spherical joint. The axes of the two prismatic joints are perpendicular to each other and each one moves along one of the axes of the Cartesian system. The third branch consists of two revolute joints, a prismatic joint and a spherical joint. In this branch, the rotation axes of the two revolute joints pass from one point.

The most famous spherical parallel robot was first studied by C.M. Gosselin [19, 20]. This spherical parallel robot has a 3-RRR structure. The branches of this robot are the same and each branch has RRR connections. All the revolute joints of this robot are placed on a hypothetical sphere. The rotational axes of all joints pass from the center of a sphere. In each branch, the revolute joint that is acts as actuator is attached to the fixed base and the remaining joints are non-actuating.

Unlike spherical serial robots, the direct kinematic problem in parallel robots is complex and difficult and an identical method for solving the direct kinematic problem cannot be presented. For this reason, the use of innovative methods which reduce the degree of equations and simplify its coefficients is of great importance. Therefore, the direct kinematic problem of these types of robots has always been of interest to various researchers. One can mention solving the direct kinematic problem of spherical parallel robots by L. Temei [21], R. Vertechy [16], H. R. Daniali [22], C.M. Gosselin [19], J. Enferadi, A. A.Tootoonchi [23, 24], J. Enferadi and A. Shahi [25].

For the first time, the solution to the direct kinematic problem of a spherical parallel robot, 3-RRR, was performed by Gosselin [19]. Subsequently, other methods have been proposed to solve the direct kinematic problem of the robot [26, 28]. Also, for the first time, a specific recombination of the 3-RRR spherical parallel robot, called Agile Eye, was suggested in [29]. Thereafter, various investigations into the Agile Eye/Wrist mechanism were also carried out by other researchers and solving the direct kinematic problem was also considered [30].

In this paper, at first, the 3-RRR spherical robot and the geometric parameters of the robot are introduced. Next, the presented methods by other researchers to solve the direct kinematic problem of the robot are evaluated. After introducing the Angle-Axis representation approach, step-by-step, the modeling of the direct kinematic problem of the 3-RRR spherical parallel robot will be considered. The use of a moving local coordinate system attached to the middle link of the first branch of the robot is one of the essential steps in the modeling process. Ultimately, this modeling leads to the formation of two trigonometric equations, which are converted into an eight-degree polynomial equation using Sylvester's elimination method. Finally, to validate the obtained equations, two case studies are presented for direct kinematics analysis of the robot.

2. Introducing the 3-RRR spherical parallel robot

According to Fig. 1, the robot consists of a spherical triangle as a fixed base, P, a moving spherical triangle as the end-effecter, M, and three branches. Each kinematic branch of this robot includes Revolute- Revolute- Revolute joints. For pure rotation of moving spherical triangle around a fixed point, all axes of the joints must pass through one point, which is the center of the sphere. Without loss of generality, the radius of the spheres that include fixed base, end-effecter, actuators and middle links can be considered equal to one. The dimensions of the fixed base spherical triangle are characterized by central angles β_i, ($\angle P_1OP_2 = \beta_1$, $\angle P_2OP_3 = \beta_2$ and $\angle P_3OP_1 = \beta_3$). The unit vector $\mathbf{u}_i$ that shows the direction of the axis of the actuator joint is defined in the direction of OP_i. The unit vector $\mathbf{v}_i$ that shows the rotational direction of the middle links is defined in the direction of OA_i. The length of the P_iA_i actuator link is defined with the central angle α_i that can be expressed as $\mathbf{u}_i^T\mathbf{v}_i = \cos\alpha_i$. The direction of revolute joints connected to spherical triangle is also shown with the vector $\mathbf{w}_i$. The length of the A_iM_i, non-actuator link, is alos shown with the central angle μ_i that can be expressed as $\mathbf{v}_i^T\mathbf{w}_i = \cos\mu_i$. The dimensions of the moving spherical triangle, $M_1M_2M_3$, are also determined by the central angles λ_i, ($\angle M_1OM_2 = \lambda_1$, $\angle M_2OM_3 = \lambda_2$ and $\angle M_3OM_1 = \lambda_3$).

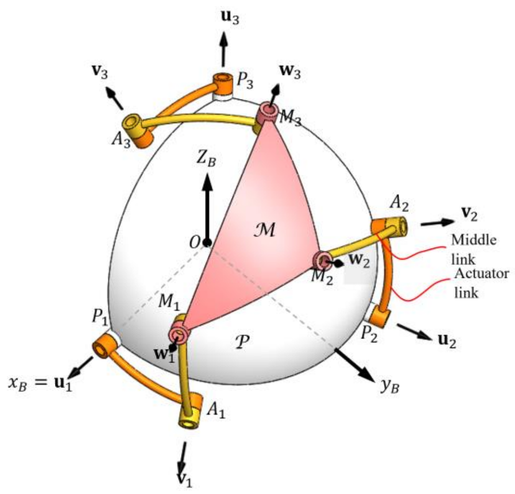

Figure 1. CAD model of a common 3-RRR spherical parallel robot

3. Problem statement

The main goal of solving the direct kinematics problem of a 3-RRR spherical robot is to obtain the orientation of the moving spherical triangle of the robot or the direction of the unit vectors $\mathbf{w}_i$. The recent methods that have been used to solve the direct kinematics of this class of robots can be categorized as follows,

a) Using three unit vectors $\mathbf{w}_i$ $(i = 1,2,3)$ as unknown:

In this method, the direct kinematic problem has nine unknowns which are components of three unit vectors in the fixed coordinate system. Therefore, nine equations are formed as follows

$$\mathbf{w}_i^T\mathbf{v}_i = \cos\mu_i \qquad for\ \ i = 1,2,3 \tag{1}$$

$$\mathbf{w}_i^T\mathbf{w}_{i+1} = \cos\lambda_i \qquad for\ \ i = 1,2,3 \tag{2}$$

$$\mathbf{w}_i^T\mathbf{w}_i = 1 \qquad for\ \ i = 1,2,3 \tag{3}$$

According to the above equations, equation (1) consists of three linear equations and equations (2) and (3) contains six equations of degree 2. Converting these equations using an eliminating method (for instance Sylvester’s or Bezout’s method) into a polynomial with one unknown parameter leads to a polynomial of degree 64 which is really difficult to solve. Moreover, solving such equations will lead to complex answers. In [26], the direct kinematic equations of the robot have been solved using this method. To solve the system of equations created by relations (1) to (3), the “fsolve” command of the MATLAB software and the initial prediction of the answers are used.

b) Using Euler’s angles of the moving spherical triangle as unknowns

In this method, a local coordinate system attaches to the moving spherical triangle. The unit vectors $\mathbf{w}_i$ are determined in local coordinate system and can be written in fixed coordinate system {B} using three Euler’s angles as follows,

$$\mathbf{w}_i = {}_M^B\mathbf{R}\,{}^M\mathbf{w}_i \qquad for\ \ i = 1,2,3 \tag{4}$$

Where ${}^M\mathbf{w}_i$ is the components of the $\mathbf{w}_i$ vectors in the local coordinate system {M} and ${}_M^B\mathbf{R}$ is the rotation matrix of the local coordinate system {M} relative to the fixed coordinate system {B}. By substituting $\mathbf{w}_i$ in equation (1), three trigonometric equations are obtained in terms of the three unknown Euler’s angles. These three trigonometric equations can be converted to a polynomial equation of degree 16. However, as suggested in

[19], local and fixed coordinate system can be considered in such a way that one of Euler's angles is equal to the central angle of the middle link. In this case, the matrix ${}_{M}^{B}\mathbf{R}$ can be written only in terms of two unknown angles. Using this method, the final polynomial degree can be reduced from 16 to 8.

c) An approach based on input/output (I/O) equations of spherical four-bar mechanism

In this method, the 3-RRR spherical parallel robot is divided into two spherical four-bar mechanisms while the output of one of these mechanisms is considered as the input of another mechanism [27]. The obtained equations are based on two unknown angles that can be determined by the orientation of the moving spherical triangle. In the paper, the values of these two unknown angles are obtained as semi-graphic. However, the paper mentioned a number of mistakes in the formulation of formulas as well as in numerical examples which were then corrected by the authors themselves [31].

In this paper, the problem of direct kinematics of the 3-RRR spherical robot will be invetigated and a new method for the general structure of this spherical robot will be presented. The main feature of the proposed method is its simplicity and the use of the angle-axis representation which will be explained in the following sections step-by-step.

4. Angle-axis representation

Before the kinematic analysis, the rotation matrix proposed by Rodriguez [32] is introduced. Rodriguez showed that a rotation matrix can be defined as follows,

$$\mathbf{R}(\mathbf{e},\theta) = \mathbf{I}_{3\times3} + (1 - \mathrm{c}\,\theta)\mathbf{E}^2 + \mathrm{s}\,\theta\,\mathbf{E} \qquad (5)$$

Where $\mathbf{e}$ is the unit vector representing the axis of the rotation and the angle θ represents the amount of this rotation around the unit vector $\mathbf{e}$. Also, the matrix $\mathbf{E}$ is an anti-symmetric matrix which is defined using the vector $\mathbf{e}$ as follows,

$$\mathbf{E} = \begin{bmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{bmatrix} \qquad (6)$$

Where e_x, e_y and e_z are the components of the unit vector $\mathbf{e}$ in the Cartesian coordinate system. Now, consider the vector $\mathbf{v}$ which is transferred by the matrix $\mathbf{R}(\mathbf{e},\theta)$. The newly transferred vector $\mathbf{v}'$ can be written as follows,

$$\mathbf{v}' = \mathbf{R}(\mathbf{e},\theta)\mathbf{v} = \mathbf{v} + (1 - \mathrm{c}\,\theta)\mathbf{E}^2\mathbf{v} + \mathrm{s}\,\theta\,\mathbf{E}\mathbf{v} \qquad (7)$$

The above equation can be written as follows,

$$\mathbf{v}' = \mathbf{v} + (1 - \mathrm{c}\,\theta)(\mathbf{e}\times(\mathbf{e}\times\mathbf{v})) + \mathrm{s}\,\theta\,(\mathbf{e}\times\mathbf{v}) \qquad (8)$$

Where $\times$ stands for external multiplication, we also know that,

$$\mathbf{e}\times(\mathbf{e}\times\mathbf{v}) = \mathbf{e}(\mathbf{e}^{\mathrm{T}}\mathbf{v}) - \mathbf{v}(\mathbf{e}^{\mathrm{T}}\mathbf{e}) = \mathbf{e}(\mathbf{e}^{\mathrm{T}}\mathbf{v}) - \mathbf{v} \qquad (9)$$

Therefore,

$$\mathbf{v}' = \mathbf{v} + (1 - \mathrm{c}\,\theta)(\mathbf{e}(\mathbf{e}^{\mathrm{T}}\mathbf{v}) - \mathbf{v}) + \mathrm{s}\,\theta\,(\mathbf{e}\times\mathbf{v}) \qquad (10)$$

Finally, it can be written,

$$\mathbf{v}' = \mathrm{c}\,\theta\,\mathbf{v} + (1 - \mathrm{c}\,\theta)\mathbf{e}\mathbf{e}^{\mathrm{T}}\mathbf{v} + \mathrm{s}\,\theta\,(\mathbf{e}\times\mathbf{v}) \qquad (11)$$

Therefore, the linear operator of the rotation matrix can be written as follows,

$$\mathbf{R}(\mathbf{e},\theta) = \mathrm{c}\,\theta\,\mathbf{I}_{3\times3} + (1 - \mathrm{c}\,\theta)\mathbf{e}\mathbf{e}^{\mathrm{T}} + \mathrm{s}\,\theta\,\mathbf{e}\times \qquad (12)$$

The above equation is a special form of the Rodriguez formula that is equivalent to the angle-axis representation [33].

5. Direct kinematic solution of the spherical parallel robot 3-RRR

In solving the direct kinematic problem of a 3-RRR spherical parallel robot, the goal is obtaining the unit vectors $\mathbf{w}_i$ in the fixed-coordinate system {B} with the known variables of the inputs (θ_i) and the constant values of β_i, α_i, μ_i and λ_i. For this purpose, the new approach to solve the direct kinematic problem of the robot is described step-by-step as follows.

Step 1: Introducing the fixed coordinate system

In the first step, the fixed coordinate system {B} is defined according to the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ (See fig. 2). Without loss of generality, we consider x_B axis in the direction of the unit vector $\mathbf{u}_1$. Therefore, the unit vector $\mathbf{u}_1$ in the fixed coordinate system is defined as,

$$\boldsymbol{x}_B \equiv \mathbf{u}_1 = [1 \quad 0 \quad 0]^T \qquad (13)$$

Axis $\boldsymbol{z}_B$ of the coordinate system {B} is perpendicular to plane OP_1P_2 using the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ as,

$$\boldsymbol{z}_B \equiv \frac{\mathbf{u}_1 \times \mathbf{u}_2}{\|\mathbf{u}_1 \times \mathbf{u}_2\|} \qquad (14)$$

The $\boldsymbol{y}_B$ axis of the fixed coordinate system {B} is also defined using the right-hand rule as,

$$\boldsymbol{y}_B \equiv \frac{\mathbf{u}_1 \times \mathbf{u}_2}{\|\mathbf{u}_1 \times \mathbf{u}_2\|} \times \mathbf{u}_1 \qquad (15)$$

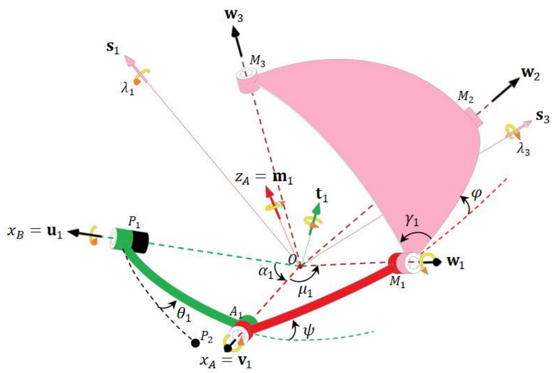

Figure 2. The unit vectors and the first branch of a 3-RRR spherical parallel robot

Step 2: Obtaining the unit vectors $\mathbf{v}_i$ in terms of rotation angles of motor

Consider the initial position of the i^{th} link as the P_iA_i arc is on the plane OP_iP_{i+1} and point A_i is on arc P_iA_i (along P_i to P_{i+1}). To obtain the unit vector $\mathbf{v}_i$, first, the vectors $\mathbf{r}_i$ and $\mathbf{t}_i$ are defined as,

$$\mathbf{r}_i = \frac{\mathbf{u}_i \times \mathbf{u}_{i+1}}{\|\mathbf{u}_i \times \mathbf{u}_{i+1}\|} \tag{16}$$

$$\mathbf{t}_i = \frac{\mathbf{u}_i \times \mathbf{v}_i}{\|\mathbf{u}_i \times \mathbf{v}_i\|} \tag{17}$$

According to figure 2, by rotating the unit vector $\mathbf{r}_i$ about the unit vector $\mathbf{u}_i$ in the positive direction as far as the angle of rotation of the motor θ_i, the unit vector $\mathbf{t}_i$ is obtained as,

$$\mathbf{t}_i = \mathbf{R}(\mathbf{u}_i, \theta_i)\mathbf{r}_i = \mathrm{c}\,\theta_i\,\mathbf{r}_i + (1 - \mathrm{c}\,\theta_i)\mathbf{u}_i\mathbf{u}_i^T\mathbf{r}_i + \mathrm{s}\,\theta_i\mathbf{u}_i \times \mathbf{r}_i \tag{18}$$

Since the unit vectors $\mathbf{u}_i$ and $\mathbf{r}_i$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{t}_i = \mathrm{c}\,\theta_i\,\mathbf{r}_i + \mathrm{s}\,\theta_i\mathbf{u}_i \times \mathbf{r}_i = [\mathrm{t}_{ix} \quad \mathrm{t}_{iy} \quad \mathrm{t}_{iz}]^T \tag{19}$$

Now, by rotating the unit vector $\mathbf{u}_i$ about the unit vector $\mathbf{t}_i$ in the positive direction as much as α_i, the unit vector $\mathbf{v}_i$ is obtained as,

$$\mathbf{v}_i = \mathbf{R}(\mathbf{t}_i, \alpha_i)\mathbf{u}_i = \mathrm{c}\,\alpha_i\,\mathbf{u}_i + (1 - \mathrm{c}\,\alpha_i)\mathbf{t}_i\mathbf{t}_i^T\mathbf{u}_i + \mathrm{s}\,\alpha_i\mathbf{t}_i \times \mathbf{u}_i \tag{20}$$

Since the vectors $\mathbf{u}_i$ and $\mathbf{t}_i$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{v}_i = \mathrm{c}\,\alpha_i\,\mathbf{u}_i + \mathrm{s}\,\alpha_i\mathbf{t}_i \times \mathbf{u}_i = [\mathrm{v}_{ix} \quad \mathrm{v}_{iy} \quad \mathrm{v}_{iz}]^T \tag{21}$$

Step 3: Defining the moving coordinate system {A}

In this step, the moving coordinate system {A}, which is connected to plane OA_1M_1, is defined. The axis $\boldsymbol{x}_A$ of this coordinate system is considered to be along the unit vector $\mathbf{v}_1$ as,

$$\boldsymbol{x}_A \equiv \mathbf{v}_1 \tag{22}$$

The axis $\boldsymbol{z}_A$ of the moving coordinate system {A} is considered to be perpendicular to plane OA_1M_1 and is defined as,

$$\boldsymbol{z}_A \equiv \mathbf{m}_1 = \frac{\mathbf{v}_1 \times \mathbf{w}_1}{\|\mathbf{v}_1 \times \mathbf{w}_1\|} \tag{23}$$

The axis $\boldsymbol{y}_A$ of the coordinate system {A} is also defined using the right-hand rule as,

$$\boldsymbol{y}_A \equiv \mathbf{m}_1 \times \mathbf{v}_1 \tag{24}$$

Step 4: Defining the unit vectors $\mathbf{v}_1$, $\mathbf{w}_1$ and $\mathbf{m}_1$ in moving coordinate system $\{A\}$

According to Fig.2, we can easily show the vectors $\mathbf{v}_1$, $\mathbf{w}_1$ and $\mathbf{m}_1$ in the moving coordinate system {A} as,

$$\boldsymbol{x}_A \equiv {}^{\mathrm{A}}\mathbf{v}_1 = [1 \ \ 0 \ \ 0]^T \tag{25}$$

$$\boldsymbol{z}_A \equiv {}^{\mathrm{A}}\mathbf{m}_1 = [0 \ \ 0 \ \ 1]^T \tag{26}$$

$${}^{\mathrm{A}}\mathbf{w}_1 = [c\mu_1 \ s\mu_1 \ \ 0]^T \tag{27}$$

Step 5: Obtaining the unit vector $\boldsymbol{s}_1$ in coordinate system $\{\boldsymbol{A}\}$

Now, by rotating the unit vector $\boldsymbol{m}_1$ about the unit vector $\mathbf{w}_1$ in the positive direction as much as φ, the unit vector $\mathbf{s}_1$ is obtained as,

$$\mathbf{s}_1 = \mathbf{R}(\mathbf{w}_1, \varphi)\mathbf{m}_1 = \mathrm{c}\,\varphi\,\mathbf{m}_1 + (1 - \mathrm{c}\,\varphi)\mathbf{w}_1\mathbf{w}_1^T\mathbf{m}_1 + \mathrm{s}\,\varphi\,\mathbf{w}_1 \times \mathbf{m}_1 \tag{28}$$

Since the vectors $\mathbf{w}_1$ and $\mathbf{m}_1$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{s}_1 = \mathrm{c}\,\varphi\,\mathbf{m}_1 + \mathrm{s}\,\varphi\,\mathbf{w}_1 \times \mathbf{m}_1 \tag{29}$$

Substituting the unit vectors $\mathbf{w}_1$ and $\mathbf{m}_1$ from (27) and (26) in the above equation, the unit vector $\mathbf{s}_1$ can be written in terms of φ in the moving coordinate system {A} as,

$${}^{A}\mathbf{s}_1 = [s\mu_1 s\varphi \quad -c\mu_1 s\varphi \quad c\varphi]^T \tag{30}$$

Step 6: Obtaining unit vector $\boldsymbol{s}_3$ in coordinate system $\{\boldsymbol{A}\}$

According to Fig.2, we can easily obtain the unit vector $\mathbf{s}_3$ by rotating the unit vector $-\boldsymbol{s}_1$ about the unit vector $\mathbf{w}_1$ as much as γ_1. Therefore, it can be written using the angle-axis representation formula as,

$$\mathbf{s}_3 = -\mathbf{R}(\mathbf{w}_1, \gamma_1)\mathbf{s}_1 = -\mathrm{c}\,\gamma_1\,\mathbf{s}_1 - (1 - \mathrm{c}\,\gamma_1)\mathbf{w}_1\mathbf{w}_1^T\mathbf{s}_1 - \mathrm{s}\,\gamma_1\,\mathbf{w}_1 \times \mathbf{s}_1 \tag{31}$$

Where $\gamma_1$is the angle between the two planes OM_1M_2 and OM_1M_3. Since the unit vectors $\mathbf{w}_1$ and $\mathbf{s}_1$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{s}_3 = -\mathrm{c}\,\gamma_1\,\mathbf{s}_1 - \mathrm{s}\,\gamma_1\,\mathbf{w}_1 \times \mathbf{s}_1 \tag{32}$$

Substituting the unit vectors $\mathbf{w}_1$ and $\mathbf{m}_1$ from (30) and (27) in the above equation, the unit vector $\mathbf{s}_3$ can be written in terms of φ in the moving coordinate system {A} as,

$${}^{A}\mathbf{s}_3 = \begin{bmatrix} -c\gamma_1 s\mu_1 s\varphi - s\gamma_1 s\mu_1 c\varphi \\ c\gamma_1 c\mu_1 s\varphi + s\gamma_1 c\mu_1 c\varphi \\ s\gamma_1 s\varphi - c\gamma_1 c\varphi \end{bmatrix} \tag{33}$$

Where γ_1 is obtained using the spherical geometry according to Appendix A.

Step 7: Obtaining the unit vector $\boldsymbol{w_2}$ in the moving coordinate system $\{\boldsymbol{A}\}$

According to Fig.2, the unit vector $\mathbf{w}_2$ can be easily obtained by rotating the unit vector $\mathbf{w}_1$ about the unit vector $\mathbf{s}_1$ as much as λ_1. Therefore, it can be written using the angle-axis representation as,

$$\mathbf{w}_2 = \mathbf{R}(\mathbf{s}_1, \lambda_1)\mathbf{w}_1 = \mathrm{c}\,\lambda_1\,\mathbf{w}_1 + (1 - \mathrm{c}\,\lambda_1)\mathbf{s}_1\mathbf{s}_1^T\mathbf{w}_1 + \mathrm{s}\lambda_1\mathbf{s}_1 \times \mathbf{w}_1 \tag{34}$$

Since the unit vectors $\mathbf{w}_1$ and $\mathbf{s}_1$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{w}_2 = \mathrm{c}\,\lambda_1\,\mathbf{w}_1 + \mathrm{s}\lambda_1\mathbf{s}_1 \times \mathbf{w}_1 \tag{35}$$

The above equation can be rewritten in the moving coordinate system {A} as,

$${}^{A}\mathbf{w}_2 = \mathrm{c}\,\lambda_1\ {}^{A}\mathbf{w}_1 + \mathrm{s}\lambda_1\ {}^{A}\mathbf{s}_1 \times {}^{A}\mathbf{w}_1 \tag{36}$$

As ${}^{\mathrm{A}}\mathbf{s}_1$is in terms of unknown angle φ, therefore, ${}^{\mathrm{A}}\mathbf{w}_2$ in the moving coordinate system {A} is also obtained in terms of the unknown parameter φ. Substituting ${}^{\mathrm{A}}\mathbf{s}_1$ and ${}^{A}\mathbf{w}_1$ from (30) and (27) in the above equation, it will be simplified as,

$${}^{\mathrm{A}}\mathbf{w}_2 = \begin{bmatrix} c\lambda_1 c\mu_1 - \mathrm{s}\lambda_1 s\mu_1 c\varphi \\ c\lambda_1 s\mu_1 + \mathrm{s}\lambda_1 c\mu_1 c\varphi \\ \mathrm{s}\lambda_1 s\varphi \end{bmatrix} \tag{37}$$

Step 8: Obtaining the unit vector $\mathbf{w}_3$ in the moving coordinate system $\{\boldsymbol{A}\}$

According to Fig.2, the unit vector $\mathbf{w}_3$ can be easily obtained by rotating the unit vector $\mathbf{w}_1$ about the unit vector $-\mathbf{s}_3$ as much as λ_3. Therefore,

$$\mathbf{w}_3 = \mathbf{R}(-\mathbf{s}_3, \lambda_3)\mathbf{w}_1 = \mathrm{c}\,\lambda_3\,\mathbf{w}_1 + (1 - \mathrm{c}\,\lambda_3)\mathbf{s}_3\mathbf{s}_3^T\mathbf{w}_1 - \mathrm{s}\,\lambda_3\mathbf{s}_3 \times \mathbf{w}_1 \tag{38}$$

Since the unit vectors $\mathbf{w}_1$ and $\mathbf{s}_3$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{w}_3 = \mathrm{c}\,\lambda_3\,\mathbf{w}_1 - \mathrm{s}\,\lambda_3\mathbf{s}_3 \times \mathbf{w}_1 \tag{39}$$

The above equation can be rewritten in the moving coordinate system {A} as,

$${}^{A}\mathbf{w}_3 = \mathrm{c}\,\lambda_3\ {}^{\mathrm{A}}\mathbf{w}_1 - \mathrm{s}\,\lambda_3\ {}^{A}\mathbf{s}_3 \times {}^{A}\mathbf{w}_1 \tag{40}$$

Since ${}^{\mathrm{A}}\mathbf{s}_3$ is in terms of unknown angle φ, therefore, ${}^{\mathrm{A}}\mathbf{w}_3$ in the moving coordinate system {A} is also obtained in terms of the unknown parameter φ. Substituting ${}^{\mathrm{A}}\mathbf{s}_3$ and ${}^{A}\mathbf{w}_1$ from (33) and (27) in the above equation, it will be simplified as,

$${}^{\mathrm{A}}\mathbf{w}_3 = \begin{bmatrix} c\lambda_3 c\mu_1 + s\lambda_3 s\mu_1 s\gamma_1 s\varphi - s\lambda_3 s\mu_1 c\gamma_1 c\varphi \\ c\lambda_3 s\mu_1 - s\lambda_3 c\mu_1 s\gamma_1 s\varphi + s\lambda_3 c\mu_1 c\gamma_1 c\varphi \\ s\lambda_3 c\gamma_1 s\varphi + s\lambda_3 s\gamma_1 c\varphi \end{bmatrix} \tag{41}$$

Step 9: Obtaining the rotation matrix ${}^{B}_{A}\mathbf{R}$ in terms of unknown angle ψ

As the values of the unit vectors $\mathbf{v}_1$ and $\mathbf{u}_1$ are known in the fixed coordinate system {B} and since the unit vector $\mathbf{t}_1$ and plane $0P_1A_1$ are perpendicular to each other, the unit vector $\mathbf{m}_1$ can be easily obtained by rotating the unit vector $\boldsymbol{t}_1$ about the unit vector $\mathbf{v}_1$ as much as ψ. Therefore, we can write,

$$\mathbf{m}_1 = \mathbf{R}(\mathbf{v}_1, \psi)\mathbf{t}_1 = \mathrm{c}\,\psi\,\mathbf{t}_1 + (1 - \mathrm{c}\,\psi)\mathbf{v}_1\mathbf{v}_1^T\mathbf{t}_1 + \mathrm{s}\,\psi\,\mathbf{v}_1 \times \mathbf{t}_1 \tag{42}$$

Since the unit vectors $\mathbf{t}_1$ and $\mathbf{v}_1$ are perpendicular to each other, the above equation will be simplified as,

$$\mathbf{m}_1 = \mathrm{c}\,\psi\,\mathbf{t}_1 + \mathrm{s}\,\psi\,\mathbf{v}_1 \times \mathbf{t}_1 \tag{43}$$

On the other hand, by replacing $\mathbf{u}_1$ from equation (12) and $\mathbf{r}_1 = [0 \quad 0 \quad 1]^T$ in equation (19), the unit vector $\mathbf{t}_1$ is given in terms of the rotational angle of the first motor θ_1 as,

$$\mathbf{t}_1 = \mathrm{c}\,\theta_1\,\mathbf{r}_1 + \mathrm{s}\,\theta_1\mathbf{u}_1 \times \mathbf{r}_1 = [0 \quad -s\theta_1 \quad c\theta_1]^T \tag{44}$$

Substituting the unit vectors $\mathbf{u}_1$ and $\mathbf{t}_1$ from (12) and (44) in equation (21), the unit vector $\mathbf{v}_1$ can be written in terms of the known parameter θ_1 as,

$$\mathbf{v}_1 = c\,\alpha_1\,\mathbf{u}_1 + s\,\alpha_1\mathbf{t}_1 \times \mathbf{u}_1 = [c\alpha_1 \quad s\alpha_1 c\theta_1 \quad s\alpha_1 s\theta_1]^T \tag{45}$$

Now, substituting the unit vectors $\mathbf{t}_1$ and $\mathbf{v}_1$ from (44) and (45) in equation (43), the unit vector $\mathbf{m}_1$ can be obtained in terms of unknown parameter ψ in the fixed coordinate system {B} as,

$$\mathbf{m}_1 = \begin{bmatrix} s\alpha_1 s\psi \\ -s\theta_1 c\psi - c\theta_1 c\alpha_1 s\psi \\ c\theta_1 c\psi - s\theta_1 c\alpha_1 s\psi \end{bmatrix} \tag{46}$$

Therefore, the unit vector $\mathbf{m}_1$ obtains in terms of ψ. On the other hand, it can be written according to equations (22), (23) and (24) as,

$${}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R} = [\mathbf{v}_1 \;\; (\mathbf{m}_1 \times \mathbf{v}_1) \;\; \mathbf{m}_1] \tag{47}$$

Therefore, ${}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R}$ will also be written in terms of the unknown parameter ψ,

$${}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R} = \begin{bmatrix} c\alpha_1 & -s\alpha_1 c\psi & s\alpha_1 s\psi \\ s\alpha_1 c\theta_1 & c\theta_1 c\alpha_1 c\psi - s\theta_1 s\psi & -s\theta_1 c\psi - c\theta_1 c\alpha_1 s\psi \\ s\alpha_1 s\theta_1 & s\theta_1 c\alpha_1 c\psi + c\theta_1 s\psi & c\theta_1 c\psi - s\theta_1 c\alpha_1 s\psi \end{bmatrix} \tag{48}$$

Step 10: Obtaining the unit vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ in the fixed coordinate system {A}

Since the unit vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are specified on the fixed coordinate system {B}, therefore, their description on {A} can be done using ${}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R}$ as,

$${}^{\mathrm{B}}\mathbf{v}_2 = {}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R}\,{}^{\mathrm{A}}\mathbf{v}_2 \;\Rightarrow\; {}^{\mathrm{A}}\mathbf{v}_2 = \left({}_{\mathrm{B}}^{\mathrm{A}}\mathbf{R}^T\right){}^{\mathrm{B}}\mathbf{v}_2 \tag{49}$$

$${}^{\mathrm{B}}\mathbf{v}_3 = {}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R}\,{}^{\mathrm{A}}\mathbf{v}_3 \;\Rightarrow\; {}^{\mathrm{A}}\mathbf{v}_3 = \left({}_{\mathrm{B}}^{\mathrm{A}}\mathbf{R}^T\right){}^{\mathrm{B}}\mathbf{v}_3 \tag{50}$$

As ${}_{\mathrm{A}}^{\mathrm{B}}\mathbf{R}$ is in terms of unknown angle ψ, therefore, ${}^{\mathrm{A}}\mathbf{v}_2$ and ${}^{\mathrm{A}}\mathbf{v}_3$ are also obtained in terms of the unknown parameter ψ.

Step 11: creating two trigonometric equations in terms of ψ and φ parameters

In this step, based on the value of the central angle of middle link μ_2 , and obtained unit vectors ${}^{A}\mathbf{w}_2$, ${}^{A}\mathbf{w}_3$ and ${}^{A}\mathbf{v}_2$, ${}^{A}\mathbf{v}_3$ in terms of unknown angle φ and ψ, respectively, two trigonometric equations can be written as,

$$\mathbf{v}_2^T\mathbf{w}_2 = c\,\mu_2 \quad \Rightarrow \quad {}^{A}\mathbf{v}_2^T\,{}^{A}\mathbf{w}_2 = c\,\mu_2 \tag{51}$$

$$\mathbf{v}_3^T\mathbf{w}_3 = c\,\mu_3 \quad \Rightarrow \quad {}^{A}\mathbf{v}_3^T\,{}^{A}\mathbf{w}_3 = c\,\mu_3 \tag{52}$$

Substituting equations (37), (43), (49) and (50), in the above equations, two trigonometric equations are obtained in terms of unknown angle φ and ψ as,

$$p_1 sin\psi + p_2 cos\psi + p_3 = 0 \tag{53}$$

$$p_4 sin\psi + p_5 cos\psi + p_6 = 0 \tag{54}$$

where,

$$p_j = q_{1j} sin\varphi + q_{2j} cos\varphi + q_{3j} \tag{55}$$

In which the coefficients q_{1j}, q_{2j} and q_{3j} are given in Appendix B.

Step 12: Convert two trigonometric equations to two polynomial equations

For converting the two trigonometric equations (53) and (54) to two polynomial equations, the following replacement must be done in equations (53) and (54).

$$sin\psi = \frac{2t}{1+t^2}\,, \;\; cos\psi = \frac{1-t^2}{1+t^2}\,, \;\; sin\varphi = \frac{2s}{1+s^2}\,, \;\; cos\varphi = \frac{1-s^2}{1+s^2} \tag{56}$$

where,

$$t = tan(\psi/2)\,, \quad s = tan(\varphi/2) \tag{57}$$

Therefore, equations (53) and (54) can be represented as,

$$E_1 t^2 + E_2 t + E_3 = 0 \tag{58}$$

$$E_4 t^2 + E_5 t + E_6 = 0 \tag{59}$$

where,

$$E_j = F_{1j} s^2 + F_{2j} s + F_{3j} \tag{60}$$

Step 13: Converting the two nonlinear polynomial equations to one polynomial

In this step, using the Sylvester's elimination method, the variable t is eliminated from equations (58) and (59). For this purpose, we multiply the Eq. (59) in E_1 and Eq. (58) in E_4 and subtract one from each other. We also multiply the Eq. (59) in E_3 and Eq. (58) in E_6 and subtract from each other. Then two recent equations can be written in the following matrix,

$$\begin{bmatrix} E_1E_5 - E_2E_4 & E_1E_6 - E_3E_4 \\ E_3E_4 - E_1E_6 & E_3E_5 - E_2E_6 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{61}$$

The above equation shows a linear equation system in terms of t and 1. Therefore, the determinants of the matrix of coefficients must be zero. Therefore, the final polynomials will be obtained from the following equation,

$$(E_1E_5 - E_2E_4)(E_3E_5 - E_2E_6) + (E_1E_6 - E_3E_4)^2 = 0 \tag{62}$$

This polynomial is based on the parameter s and is of order eight, which is written in the following general form.

$$\sum_{p=0}^{8} N_p s^p = 0 \tag{63}$$

in which its coefficients are based on the fixed geometric parameters of the robot and variable parameters of the robot motors.

Step 14: Obtaining the two angles φ, ψ and the unit vectors $\mathbf{w}_i$

By solving Eq. (63), we can obtain the value of its corresponding angle φ using the values obtained for the parameter s as,

$$\varphi = atan2(2s\,, 1 - s^2) \tag{64}$$

By specifying the values of φ, we can calculate the vectors ${}^{A}\mathbf{w}_i$ according to the previous equations. Also, by inserting the value of φ in equation (53) and (54), we can easily obtain the values $sin\psi$ and $cos\psi$ and compute the value of the angle ψ as,

$$\psi = atan2(sin\psi\,, cos\psi) \tag{65}$$

By calculating ψ, we can also compute the rotation matrix ${}^{B}_{A}\mathbf{R}$ using Eq. (48). Finally, the unit vectors $\mathbf{w}_i$ is obtained in the fixed coordinate system {B} and solving the direct kinematic problem of the robot will accomplish as follows,

$$\mathbf{w}_i = {}^{B}_{A}\mathbf{R}\,{}^{A}\mathbf{w}_i \tag{66}$$

6. Case study

In this section, by representing two examples, the solutions of the direct kinematic problem of a 3-RRR spherical parallel robot are obtained and it is shown that the proposed method is an optimal method for solving the direct kinematic problem of the robot. In direct kinematics problem, the constant values of β_i, α_i, μ_i and λ_i represent the geometry of the fixed spherical triangles, the actuated links, the middle links, and the moving spherical triangles, respectively. Also, the variable parameters of the actuators θ_i are known as inputs. To determine the orientation of the moving spherical triangle as previously mentioned, two angles ψ and φ must be obtained. In the following examples, details of the calculation will be described.

6.1. Case study 1

Consider the angles of the fixed spherical triangle are equal to $\beta_1 = \beta_2 = \beta_3 = 110°$. Therefore, the unit vectors $\mathbf{u}_i$ in the fixed coordinate system {B} will be as,

$$\mathbf{u}_1 = [1 \quad 0 \quad 0]^T$$
$$\mathbf{u}_2 = [-0.3420201431 \quad 0.9396926209 \quad 0]^T$$
$$\mathbf{u}_3 = [-0.34202014 \quad -0.48845538 \quad 0.80276619]^{\mathrm{T}}$$

Using the above vectors, the unit vectors $\mathbf{r}_i$ are also calculated in the fixed coordinate system {B} as,

$$\mathbf{r_1} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]^{T}$$
$$\mathbf{r_2} = [\mathbf{0.802766191} \quad \mathbf{0.292182998} \quad \mathbf{0.519803363}]^{T}$$
$$\mathbf{r_3} = [\mathbf{0} \quad \mathbf{0.8542859370} \quad \mathbf{0.5198033639}]^{T}$$

Consider value of the motor rotation as $\theta_1 = \theta_2 = \theta_3 = 15°$ and using the derived vectors obtained above, we can calculate the vectors $\mathbf{t}_i$ as,

$$\mathbf{t_1} = [\mathbf{0} \quad -\mathbf{0.2588190451} \quad \mathbf{0.9659258263}]^{T}$$
$$\mathbf{t_2} = [\mathbf{0.90183415} \quad \mathbf{0.32824078} \quad \mathbf{0.28098602}]^{\mathrm{T}}$$
$$\mathbf{t_3} = [-\mathbf{0.24321034} \quad \mathbf{0.87119053} \quad \mathbf{0.42646896}]^{\mathrm{T}}$$

Next, consider the length of the actuated links as $\alpha_1 = \alpha_2 = \alpha_3 = 70°$. The unit vectors $\mathbf{v}_i$ are obtained as,

$$\mathbf{v_1} = [\mathbf{0.34202014} \quad -\mathbf{0.90767337} \quad \mathbf{0.24321034}]^{\mathrm{T}}$$
$$\mathbf{v_2} = [-\mathbf{0.36509468} \quad \mathbf{0.23108663} \quad \mathbf{0.90183415}]^{\mathrm{T}}$$
$$\mathbf{v_3} = [\mathbf{0.73595619} \quad -\mathbf{0.12065949} \quad \mathbf{0.66619049}]^{T}$$

Also, consider the values $\lambda_1 = \lambda_2 = \lambda_3 = 70°$ for the moving spherical triangle and using spherical geometry in accordance with Appendix A, the angle $\gamma_1 = 75.24°$ will be obtained. Finally, substituting the values $\mu_1 = \mu_2 = \mu_3 = 80°$ for the middle links and using the above obtained parameters, we can calculate the coefficients of the polynomial of Eq. (63) as presented shown in Table 1. Therefore, we can obtain the values of the angles φ and ψ from relations (64) and (65) easily as shown in Table 2. Finally, according to step 14, we can obtain the unit vectors $\mathbf{w}_i$ as shown in Table 3. The assembly modes related to these eight solutions are also shown in Figures 3 through 10.

Since in this example, for the polynomials of degree eight of equation (63), eight real solutions were obtained, therefore, the polynomial degree obtained in equation (63) can be said to be minimal and optimal.

Table 1. The obtained coefficients N_p (p=0-8) for case study 1

Coefficients	Value	Coefficients	Value
N_0	0.3482	N_5	8.5128
N_1	0.2918	N_6	-8.8615
N_2	-9.8783	N_7	0.8485
N_3	-13.0312	N_8	0.0177
N_4	17.9584		

Table 2. Values of the angles $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ for case study 1

Configuration No. (Assembly modes)	φ	ψ
1	−96.3132°	−148.0752°
2	−23.8573°	141.7683°
3	164.9933°	118.6045°
4	102.7414°	−97.5969°

5	119.7667°	74.6923°	8	21.0732°	−18.6279°
6	−177.9814°	−69.1063°			
7	−51.3827°	51.5286°			

Table 3. The assembly modes $\mathbf{w}_i$ for case study 1

Configuration No. (Assembly modes)	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$
1	$[0.8448 \quad 0.0163 \quad -0.5348]^T$	$[0.7736 \quad -0.2678 \quad 0.5743]^T$	$[0.2829 \quad -0.9333 \quad -0.2210]^T$
2	$[0.7863 \quad -0.2557 \quad 0.5624]^T$	$[-0.1314 \quad -0.9179 \quad 0.3745]^T$	$[0.5735 \quad -0.6553 \quad -0.4916]^T$
3	$[0.5024 \quad -0.2219 \quad 0.8356]^T$	$[0.6074 \quad 0.7557 \quad 0.2448]^T$	$[-0.3804 \quad 0.5079 \quad 0.7729]^T$
4	$[0.1817 \quad 0.3673 \quad -0.9122]^T$	$[-0.7262 \quad 0.6347 \quad -0.2641]^T$	$[0.3274 \quad 0.9423 \quad 0.0697]^T$
5	$[-0.1849 \quad -0.0023 \quad 0.9828]^T$	$[0.8533 \quad 0.1137 \quad 0.5089]^T$	$[0.0610 \quad 0.9303 \quad 0.3617]^T$
6	$[-0.2706 \quad 0.5118 \quad -0.8154]^T$	$[0.3075 \quad 0.9487 \quad 0.0739]^T$	$[0.7939 \quad 0.1491 \quad -0.5894]^T$
7	$[-0.5163 \quad 0.1605 \quad 0.8412]^T$	$[-0.9738 \quad -0.1609 \quad -0.1605]^T$	$[-0.2737 \quad -0.8724 \quad 0.4050]^T$
8	$[-0.8175 \quad 0.5473 \quad -0.1790]^T$	$[-0.8120 \quad -0.5836 \quad 0.0134]^T$	$[-0.5092 \quad 0.1420 \quad 0.8489]^T$

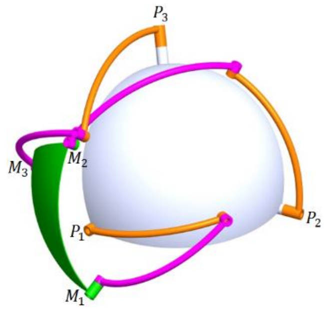

Figure 3. Assembly mode No.1 of case study 1

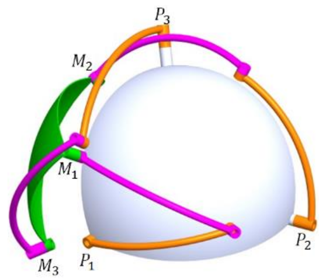

Figure 4. Assembly mode No.2 of case study 1

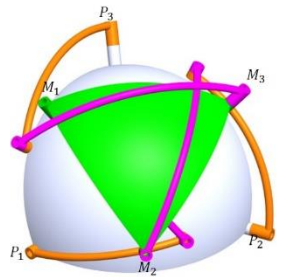

Figure 5. Assembly mode No.3 of case study 1

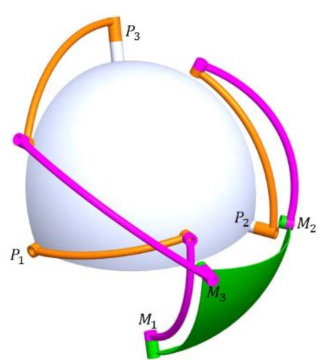

Figure 6. Assembly mode No.4 of case study 1

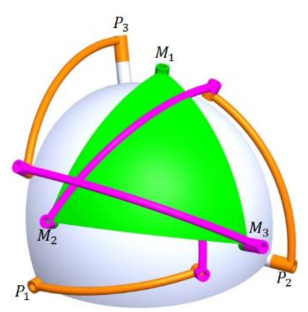

Figure 7. Assembly mode No.5 of case study 1

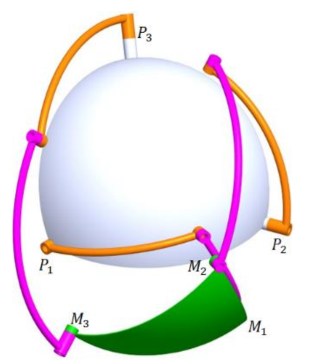

Figure 8. Assembly mode No.6 of case study 1

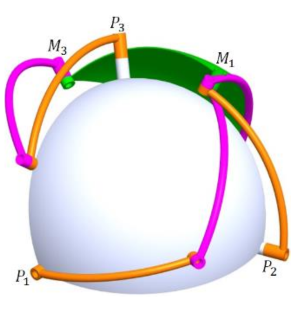

Figure 9. Assembly mode No.7 of case study 1

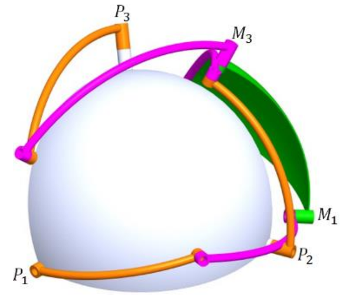

Figure 10. Assembly mode No.8 of case study 1

6.2. Case study 2

In this example, a particular case of a 3-RRR spherical parallel robot is investigated. Consider the angles of the fixed triangle are equal to $\beta_1 = \beta_2 = \beta_3 = 0°$. Thus, the unit vectors $\mathbf{u}_i$ in the fixed coordinate system {B} will be defined as,

$$\mathbf{u_1} = \mathbf{u}_2 = \mathbf{u}_3 = [1 \quad 0 \quad 0]^T$$

Using the above unit vectors, the vectors $\mathbf{r}_i$ are also calculated on the fixed coordinate system {B} as,

$$\mathbf{r_1} = \mathbf{r_2} = \mathbf{r_3} = [0 \quad 0 \quad 1]^{\mathrm{T}}$$

Consider values of the motors rotation as $\theta_1 = 0°$, $\theta_2 = 120°$, $\theta_3 = 240°$ and using the above obtained unit vectors, we can calculate the vectors $\mathbf{t}_i$ as follows,

$$\mathbf{t_1} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{1}]^{\boldsymbol{T}}$$
$$\mathbf{t_2} = [\mathbf{0} \quad -\mathbf{0.8660254035} \quad -\mathbf{0.5}]^{\boldsymbol{T}}$$
$$\mathbf{t_3} = [\mathbf{0} \quad \mathbf{0.8660254034} \quad -\mathbf{0.5}]^{\boldsymbol{T}}$$

Next, consider the length of the actuated links as $\alpha_1 = 80°$, $\alpha_2 = 50°$, $\alpha_3 = 60°$. Therefore, the unit vectors $\mathbf{v}_i$ are obtained as,

$$\mathbf{v_1} = [\mathbf{0.1736481776} \quad \mathbf{0.984807753} \quad \mathbf{0}]^{\boldsymbol{T}}$$
$$\mathbf{v_2} = [\mathbf{0.64278760} \quad -\mathbf{0.38302222} \quad \mathbf{0.66341394}]^{\boldsymbol{T}}$$
$$\mathbf{v_3} = [\mathbf{0.5} \quad -\mathbf{0.4330127025} \quad -\mathbf{0.7499999996}]^{\boldsymbol{T}}$$

Also, consider the values $\lambda_1 = \lambda_2 = \lambda_3 = 107.6°$ for the moving spherical triangle and using spherical geometry according to appendix A, the angle $\gamma_1 = 115.67°$ will be obtained. Finally, considering the values $\mu_1 = 85°$, $\mu_2 = 90°$, $\mu_3 = 100°$ for the middle links and utilizing the previously obtained parameters, we can calculate the coefficients of the polynomial of Eq. (63) as presented in Table 4. Therefore, we can obtain the values of the angles φ and ψ from equations (64) and (65) easily as shown in Table 5. Finally, according to step 14, we can obtain the unit vectors $\mathbf{w}_i$ as shown in Table 6. The assembly modes (configurations) related to these eight solutions are also shown in Figures 11 through 18.

Table 4. The obtained coefficients N_p (p=0-8) for case study 2

Coefficients	Value	Coefficients	Value
N_0	0.2095	N_5	-21.2431
N_1	-1.1886	N_6	-5.8099
N_2	-9.2843	N_7	0.6965
N_3	16.4866	N_8	0.3481
N_4	16.8635		

Table 5. Values of the angles φ and ψ for case study 2

Configuration No. (Assembly modes)	φ	ψ
1	148.0741°	80.0589°
2	−21.9495°	25.1089°
3	−80.7885°	−23.6053°
4	84.1511°	−62.5979°
5	12.1328°	−153.2741°
6	−160.1502°	−145.3424°
7	−145.4095°	−170.6340°
8	127.8862°	61.2996°

Table 6. The assembly modes $\mathbf{w}_i$ for case study 2

Configuration No. (Assembly modes)	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$
1	$[0.9962 \quad 0.0150 \quad 0.0858]^T$	$[-0.2307 \quad 0.3532 \quad 0.9067]^T$	$[-0.2921 \quad -0.9357 \quad -0.0352]^T$
2	$[0.9962 \quad 0.0789 \quad -0.0370]^T$	$[-0.3783 \quad 0.6225 \quad -0.6851]^T$	$[-0.2985 \quad 0.3240 \quad 0.8986]^T$
3	$[0.9962 \quad 0.0799 \quad 0.0349]^T$	$[-0.3145 \quad 0.4619 \quad -0.8118]^T$	$[-0.3694 \quad 0.4715 \quad 0.8008]^T$
4	$[0.9962 \quad 0.0401 \quad 0.0774]^T$	$[-0.3097 \quad -0.8096 \quad 0.4987]^T$	$[-0.2231 \quad -0.1362 \quad -0.9652]^T$
5	$[0.9962 \quad -0.0778 \quad 0.0392]^T$	$[-0.3824 \quad 0.8958 \quad 0.2265]^T$	$[-0.2503 \quad 0.2047 \quad 0.9463]^T$
6	$[0.9962 \quad -0.0717 \quad 0.0496]^T$	$[-0.2231 \quad 0.9404 \quad 0.2565]^T$	$[-0.3605 \quad 0.1558 \quad 0.9197]^T$
7	$[0.9962 \quad 0.0860 \quad 0.0142]^T$	$[-0.2328 \quad 0.8854 \quad 0.4023]^T$	$[-0.3733 \quad -0.7104 \quad 0.5967]^T$
8	$[0.9962 \quad -0.0535 \quad -0.0688]^T$	$[-0.3411 \quad 0.3962 \quad 0.8525]^T$	$[-0.2183 \quad 0.6383 \quad 0.7382]^T$

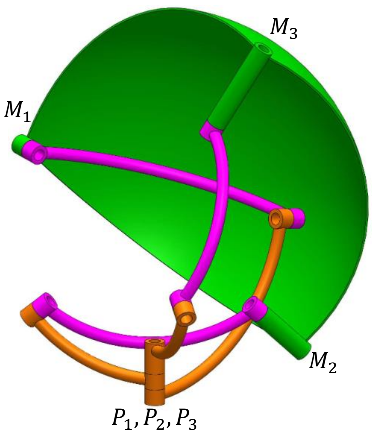

Figure 11. Assembly mode No.1 of case study 2

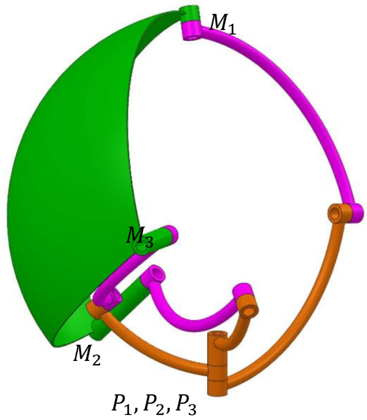

Figure 12. Assembly mode No.2 of case study 2

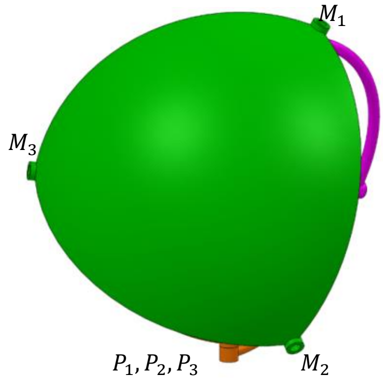

Figure 13. Assembly mode No.3 of case study 2

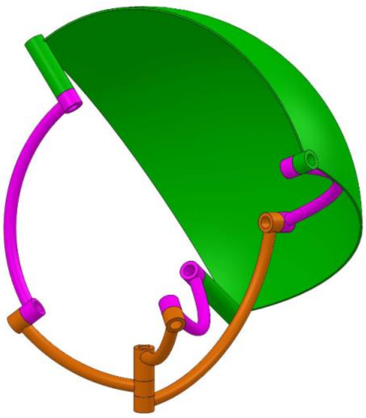

Figure 14. Assembly mode No.4 of case study 2

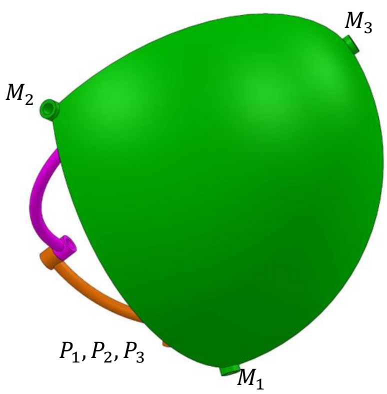

Figure 15. Assembly mode No.5 of case study 2

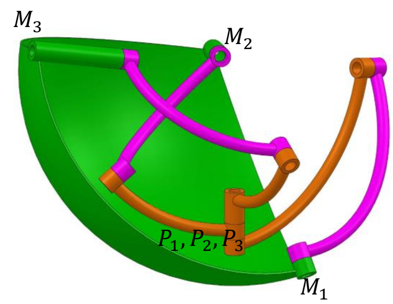

Figure 16. Assembly mode No.6 of case study 2

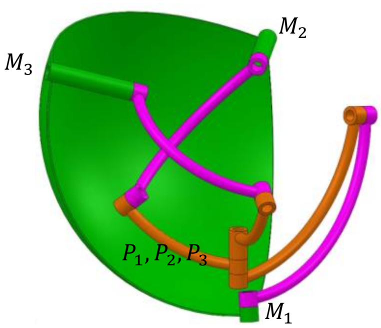

Figure 17. Assembly mode No.7 of case study 2

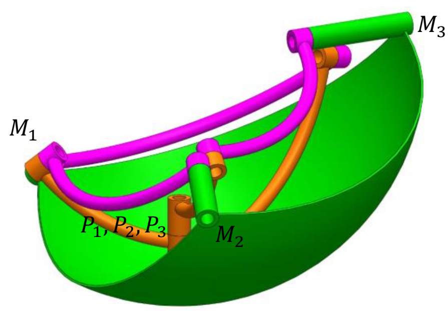

Figure 18. Assembly mode No.8 of case study 2

7. Conclusion

In this paper, first, the 3RRR spherical parallel robot was first introduced and then different methods of solving the direct kinematics of this class of robots were evaluated. A new approach for modeling the direct kinematic problem of the robot and its solution was presented. In this new approach, the angle-axis representation is used to model the direct kinematic problem which has not been used in any of the previous methods. The advantage of the new method over the previous methods is the simplicity of problem modeling based on the structure of the robot which leads to the coefficients of the final equations (Appendix B). Using this approach, the direct kinematics problem of two sample robots was solved and eight different configurations were obtained for each one. The configurations were shown graphically by a commercial modeling software package which confirms the accuracy of the proposed model. It can also be concluded that the obtained polynomial degree is minimal and the proposed modeling method is also optimal. The advantage of the proposed method was the use of two evident geometric angles in solving the direct kinematic problem of the robot. By using the presented analytical geometric method, the direct kinematic analysis of this class of robots, which are inherently very complex, has become understandable and conceptual.

8. Appendices

8.1. Appendix A: Calculating the angel $\boldsymbol{\gamma_1}$

$$g = \sum_{i=1}^{3} \lambda_i \Big/ 2$$

$$f = \left(\prod_{i=1}^{3} \sin(g - \lambda_i) \Big/ \sin g \right)^{0.5}$$

$$\gamma_1 = 2 \tan^{-1}(f/\sin(g - \lambda_2))$$

8.2. Appendix B: Coefficients $\boldsymbol{q_{1j}}$, $\boldsymbol{q_{2j}}$ and $\boldsymbol{q_{3j}}$

$$q_{11} = s\lambda_1 s\alpha_1 v_{2x} - s\lambda_1 c\alpha_1 c\theta_1 v_{2y} - s\lambda_1 c\alpha_1 s\theta_1 v_{2z}$$
$$q_{21} = v_{2z} c\theta_1 s\lambda_1 c\mu_1 - v_{2y} s\theta_1 s\lambda_1 c\mu_1$$
$$q_{31} = v_{2z}\, c\theta_1 c\lambda_1 s\mu_1 - v_{2y} s\theta_1 c\lambda_1 s\mu_1$$
$$q_{12} = s\lambda_1 v_{2z} c\theta_1 - s\lambda_1 v_{2y} s\theta_1$$
$$q_{22} = v_{2y} c\alpha_1 c\theta_1 s\lambda_1 c\mu_1 - s\alpha_1 v_{2x} s\lambda_1 c\mu_1 + v_{2z} c\alpha_1 s\theta_1 s\lambda_1 c\mu_1$$
$$q_{32} = -s\alpha_1 v_{2x}\, c\lambda_1 s\mu_1 + v_{2y} c\alpha_1 c\theta_1 c\lambda_1 s\mu_1 + v_{2z} c\alpha_1 s\theta_1 c\lambda_1 s\mu_1$$
$$q_{13} = 0$$
$$q_{23} = -c\alpha_1 v_{2x}\, s\lambda_1 s\mu_1 - s\alpha_1 c\theta_1 v_{2y}\, s\lambda_1 s\mu_1 - s\alpha_1 s\theta_1 v_{2z}\, s\lambda_1 s\mu_1$$
$$q_{33} = c\alpha_1 v_{2x}\, c\lambda_1 c\mu_1 + s\alpha_1 c\theta_1 v_{2y}\, c\lambda_1 c\mu_1 - c\mu_2 + s\alpha_1 s\theta_1 v_{2z}\, c\lambda_1 c\mu_1$$
$$q_{14} = v_{3y}\, s\theta_1 s\lambda_3 c\mu_1 s\gamma_1 - v_{3y}\, c\alpha_1 c\theta_1 s\lambda_3 c\gamma_1 - v_{3z}\, c\alpha_1 s\theta_1 s\lambda_3 c\gamma_1 - v_{3z}\, c\theta_1 s\lambda_3 c\mu_1 s\gamma_1 + s\alpha_1 v_{3x}\, s\lambda_3 c\gamma_1$$
$$q_{24} = -v_{3y}\, s\theta_1 s\lambda_3 c\mu_1 c\gamma_1 + v_{3z}\, c\theta_1 s\lambda_3 c\mu_1 c\gamma_1 - v_{3y}\, c\alpha_1 c\theta_1 s\lambda_3 s\gamma_1 - v_{3z}\, c\alpha_1 s\theta_1 s\lambda_3 s\gamma_1 + s\alpha_1 v_{3x}\, s\lambda_3 s\gamma_1$$
$$q_{34} = v_{3z}\, c\theta_1 c\lambda_3 s\mu_1 - v_{3y}\, s\theta_1 c\lambda_3 s\mu_1$$
$$q_{15} = s\alpha_1 v_{3x}\, s\lambda_3 c\mu_1 s\gamma_1 - v_{3y}\, c\alpha_1 c\theta_1 s\lambda_3 c\mu_1 s\gamma_1 - v_{3z}\, c\alpha_1 s\theta_1 s\lambda_3 c\mu_1 s\gamma_1 + v_{3z}\, c\theta_1 s\lambda_3 c\gamma_1 - v_{3y}\, s\theta_1 s\lambda_3 c\gamma_1$$
$$q_{25} = -v_{3y}\, s\theta_1 s\lambda_3 s\gamma_1 + v_{3y}\, c\alpha_1 c\theta_1 s\lambda_3 c\mu_1 c\gamma_1 - s\alpha_1 v_{3x}\, s\lambda_3 c\mu_1 c\gamma_1 + v_{3z}\, c\alpha_1 s\theta_1 s\lambda_3 c\mu_1 c\gamma_1 + v_{3z}\, c\theta_1 s\lambda_3 s\gamma_1$$
$$q_{35} = -s\alpha_1 v_{3x}\, c\lambda_3 s\mu_1 + v_{3y}\, c\alpha_1 c\theta_1 c\lambda_3 s\mu_1 + v_{3z}\, c\alpha_1 s\theta_1 c\lambda_3 s\mu_1$$
$$q_{16} = s\alpha_1 s\theta_1 v_{3z}\, s\lambda_3 s\mu_1 s\gamma_1 + s\alpha_1 c\theta_1 v_{3y} s\lambda_3 s\mu_1 s\gamma_1 + c\alpha_1 v_{3x} s\lambda_3 s\mu_1 s\gamma_1$$
$$q_{26} = -c\alpha_1 v_{3x} s\lambda_3 s\mu_1 c\gamma_1 - s\alpha_1 s\theta_1 v_{3z} s\lambda_3 s\mu_1 c\gamma_1 - s\alpha_1 c\theta_1 v_{3y} s\lambda_3 s\mu_1 c\gamma_1$$
$$q_{36} = c\alpha_1 v_{3x} c\lambda_3 c\mu_1 - c\mu_3 + s\alpha_1 s\theta_1 v_{3z} c\lambda_3 c\mu_1 + s\alpha_1 c\theta_1 v_{3y} c\lambda_3 c\mu_1$$

List of symbols

A_i	The joints of the middle links	$\mathbf{r}_i$	The unit vector perpendicular to the plane crossing the $\mathbf{u}_i$ and $\mathbf{u}_{i+1}$ vectors
$\{A\}$	Moving coordinate system	$\mathbf{t}_i$	The unit vector perpendicular to the plane crossing the $\mathbf{u}_i$ and $\mathbf{v}_i$ vectors
$\{B\}$	Fixed coordinate system	$\mathbf{u}_i$	The axis direction of the active joints
$\mathbf{E}$	An anti-symmetric matrix	$\mathbf{v}_i$	The axis direction of the joints of the middle links
$\mathbf{m}_i$	The unit vector perpendicular to the plane crossing the $\mathbf{v}_i$ and $\mathbf{w}_i$ vectors	$\mathbf{w}_i$	The axis direction of the joints connected to the moving platform
M_i	The joints connected to the moving platform	$\{\boldsymbol{x}_A, \boldsymbol{y}_A, \boldsymbol{z}_A\}$	The unit vectors of the moving coordinate system
M	A spherical triangle as the moving platform	$\{\boldsymbol{x}_B, \boldsymbol{y}_B, \boldsymbol{z}_B\}$	The unit vectors of the fixed coordinate system
$\{M\}$	Local coordinate system	α_i	The angle of the actuator links
N_p	polynomial coefficients of the direct kinematic equation	β_i	The dimensions of the fixed base spherical triangle
P	A spherical triangle as the fixed base	θ_i	Input joint angles
P_i	The active joints	λ_i	The dimensions of the moving platform spherical triangle

${}^{B}_{M}\mathbf{R}$	The rotation matrix of the local coordinate system relative to the fixed coordinate system	μ_i	The angle of the middle links

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