INTERNATIONAL JOURNAL OF ROBOTICS
(THEORY AND APPLICATIONS)
E-mail: IJR@kntu.ac.ir

Asymptotic Boundary Feedback Stabilization of a Flexible Gantry Manipulator with One Axis of Rotation

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ARTICLE INFO

Article history:

Received: February 20, 2019.

Received in revised form: December 27, 2019.

Accepted: December 30, 2019.

Keywords:

Flexible gantry robot manipulator,

Distributed parameter system,

Vibration dissipation,

Boundary control

ABSTRACT

The main purpose of this paper is to design a novel boundary control method grounded in the Hamilton principle in order to control a gantry robot manipulator with one axis of rotation while the link’s transverse vibrations were examined and the system is affected by rigid body nonlinear large rotation and translation. At the beginning, partial differential equations (PDE) and ordinary differential equations (ODE) will be derived as a governing equation predicated on Hamilton's principle. In this paper, the control rules were: first, the system changes its position to the desired position. Second, overcoming the flexible link transverse vibrations, and finally, controlling angular position. Based on Lyapunov functions and considering external boundary disturbance, suitable control feedback signals and boundary disturbance observer are proposed to achieve control rules that were mentioned above and reduce external boundary disturbance impacts at the same time. Eventually, it was proved that by choosing appropriate design parameters, system states and position error converge exponentially to a small neighborhood of zero. To show the performance of our control method, numerical simulation outcomes are dispensed.

1. Introduction

Controlling flexible manipulator’s motion has perpetual demand for researchers [1]. The critical concerns that these systems will face are dynamic accuracy, higher operating speed, and ensuring operating safety. For overhauling the efficiency, it is best to use lightweight links, however, for high-speed precision transportation systems, unwanted vibration because of the manipulator link's flexibility and external disturbances lead to inaccurate positioning that is time-consuming and risky as well so this was the reason that controlling methods for reducing vibration and precise positioning are demandable and lots of publications were published about techniques for regulating Infinite-Dimensional structures [6], [17], [21].

The complexity of control design surges as a result of the representation of the flexible body as a distributed parameter system with infinite degrees of freedom.

Control techniques such as assumed-mode, lumped-parameter, and finite element method are used to regulate the end effector and suppress vibration at the same time and all of them depend on discretizing the PDE model into a set of ODEs.

The methods mentioned above don't consider infinite number of vibration modes, so to avoid higher-order controlling problems and spillover observation and control occurrence, a new control PDE-based method that doesn't need discretization for flexible links has been established.

The boundary control method's advantage is its implementation which makes this method widely functional in any control strategies for systems governed by PDEs so this method appears to be the most practical method among infinite-dimensional control methods [3].

In [22], [26], [31] researchers invented boundary controllers to make certain that closed-loop stability for an unlimited number of modes which neglect the spillover occurrence, occur for non-discretized PDE models.

The boundary control method removes the in-domain sensing/actuating issue. This means that the controllers basically acknowledge apparent physical features.

Moreover, for different kinds of flexible systems including strings [5], [12], [17] container cranes [16], [8], composite plates [25], [29] and composite shells containing fluid [24] mentioned researchers have presented some boundary controllers for each one.

Originally, the boundary control method was conducted by researchers to deal with challenges that they were facing in controlling flexible manipulators.

One of the basic flexible manipulators is beams, [23] used control torque and force to provide a new vibration reduction technique. The author of [20] examined one of the beams issues and used (TTS) two-time scale control theory and the boundary control technique to deal with that. Vibration stabilization depending on the Lyapunov function to consider boundary output restrictions was managed in [15]. In [9] controlling vibration with input saturation was studied.

The writers of [31] asymptotically controlled a flexible arm with two-dimensional rigid body rotation. Furthermore, in [19] a controller was devised for a nonlinear 3-dimensional flexible arm while gravitational energy was neglected. For neutralizing unknown boundary disturbances, disturbance observers were used. Disturbance observer of translating beam was used in [18] and as a result, the disturbance's value of the bonds was assessed.

Designing boundary disturbances in order to estimate the time-varying boundary disturbances, was inspected in [14] and [15].

In this article, for proposed gantry flexible manipulator systems that can be seen in Figure 1, a novel boundary control method has been innovated.

In this work, the transverse vibration and rotation angle have been controlled while the suggested manipulator has moved to its demanded position. Furthermore, for this system, a novel disturbance observer has been presented.

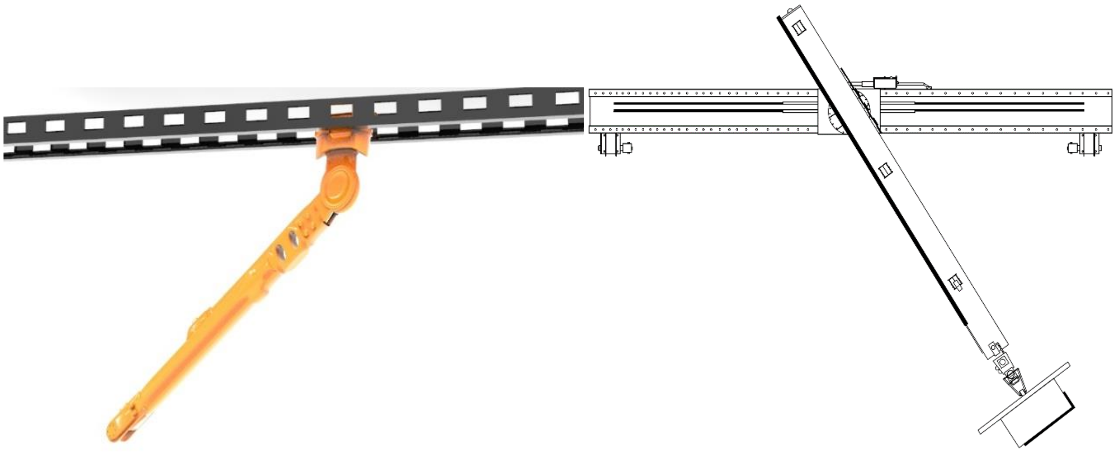

Figure 1. Proposed gantry manipulator systems

In previous published papers, they have implemented boundary control methods grounded in linear flexible arms in the vertical plane. Moreover, designing control strategy was barely grounded in the original hybrid PDE-ODE dynamic modeling and also they didn't consider the effect of gravitational force and base position tracking to the desired position that neglecting them could affect controller performances. These guesses might be suitable for small-angle regulation and slow-motion of the manipulator's base specifically near the set point. In many applications like in the vertical plane and in the presence of time-varying boundary disturbances, gaining the exact end-effector location by considering all issues is inevitable.

It is worth saying that, the control designing and stability procedures are based on original hybrid PDE-ODE proposed governing equations without any simplifications in order to overcome accurate positioning. The main contributions of this research are summarized as follows:

i. The original PDE-ODE equations of motion have been derived from Hamiltonian method and in the dynamical model everything that has a huge impact on the control design strategy like nonlinearities, gravitational force, flexibility, the payload, base mass dynamics, coupling effects between the large rotational angle dynamics and changeable boundary disturbances was concerned.

ii. For the proposed system which at the same time, base accurate positioning, vibrations restraint, and huge nonlinear angular rotation were examined while any simplifications and spillover were barely neglected, a unique boundary control technique has been presented.

iii. Based on controlling purposes and rules, to provide the exponential stability of the closed-loop system while there are no boundary disturbances, a nonlinear Lyapunov design has been effectively used. Moreover, by utilizing boundary observer, uniform boundedness of the closed-loop system, which is affected by the unknown time-varying boundary disturbance, is gained. Furthermore, the boundedness of all closed-loop signals is also expressed. These signals include the base's horizontal displacement, boundary deflection, rotational angle, and shear measurement with their derivatives which make them feasible.

This paper is organized as follows. In section 2 the system kinematics and dynamics are proposed. In section 3 some preliminaries are examined. Section 4 includes designing boundary control rules by using the Lyapunov method and as a result, a complex and proper Lyapunov functional will be adopted. Furthermore, grounded in over mentioned control inputs, the exponential stability and uniform boundedness of the closed-loop system in the absence and existence of boundary disturbance have been examined, respectively. Section 5 consists of simulation results and is arranged to show the effectiveness of the boundary control approach. Finally, in section 6 the summarized conclusion can be seen.

2. Mathematical Modeling

The simplified diagram of the proposed system including a portable base, flexible arm, and payload at the bottom, can be seen in figure 2. Frame XOY is the fixed inertial frame, and the motion of the system takes place in a vertical plane. Consider the spatial coordinate along the longitude of the flexible link is x, t denotes time, $w(x, t)$ represents the transverse displacement because of lateral vibrations of the flexible link at the spatial coordinate and time, $\theta(t)$ represents the angle of rotation and $\dot{\theta}(t)$, $\ddot{\theta}(t)$ are first-order derivatives and second-order derivatives respect to time. Consider $\eta(t)$ and $\dot{\eta}(t)$, trolley position and velocity respectively. Furthermore, let m1 and m2 be the equivalent mass of base and payload respectively, l denotes the length of the link, ρ the mass per unit of length and the subscripts x; t denotes the partial derivatives with respect to x; t, respectively.

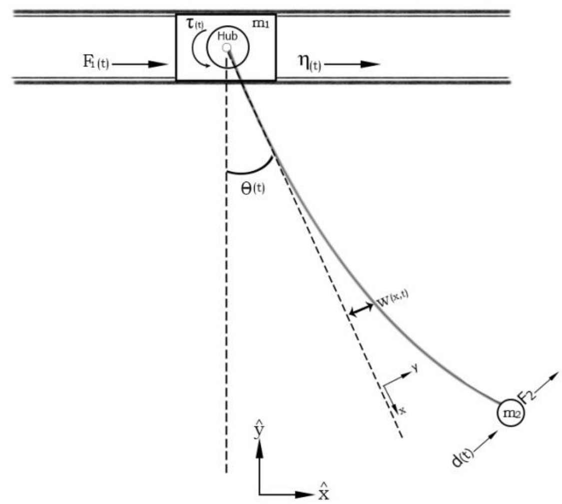

Fig 2. Schematic of flexible gantry manipulator with input forces and boundary disturbance

The kinetic energy of the Gantry flexible manipulator system can be represented as:

$$K_T = 1/2 m_1 \dot{\eta}^2 + 1/2\rho \int_0^{l(t)} \left(v_x{}^2 + v_y{}^2\right) dx + 1/2 m_2 \left(v_{xl}{}^2 + v_{yl}{}^2\right) \tag{1}$$

v_x and v_y denote the velocity vectors of flexible system in x and y directions and can be defined as:

$$v_x = \dot{\eta} + x\dot{\theta}\cos(\theta) + (w_t(x,t))\cos(\theta) - w(x,t)\dot{\theta}\sin(\theta) \tag{2}$$

$$v_y = x\dot{\theta}(t)\sin(\theta(t)) + (w_t(x,t))\sin(\theta(t)) - w(x,t)\dot{\theta}(t)\cos(\theta(t)) \tag{3}$$

Using equations (1) ,(2) and (3) the kinetic energy can be rewritten as:

$$
\begin{aligned}
K_T &= 1/2 m_1 \dot{\eta}^2 \\
&+1/2\rho\int_0^l \left\{\begin{aligned}&\dot{\eta}(t)^2+\left(x\,\dot{\theta}(t)\right)^2\\&+\left(w_t(x,t)\right)^2\\&+\left(w(x,t)\dot{\theta}(t)\right)^2\end{aligned}\right\}dx \\
&+1/2\rho\int_0^l \{2\dot{\eta}(t)\dot{\theta}(t)x\cos\left(\theta(t)\right) \\
&+2\dot{\eta}(t)\left(w_t(x,t)\right)\cos\left(\theta(t)\right) \\
&-2\dot{\eta}(t)\dot{\theta}(t)w(x,t)\sin\left(\theta(t)\right) \\
&+2\left(w_t(x,t)\right)\dot{\theta}(t)x\}dx \\
&+1/2m_2\{\dot{\eta}(t)^2+\left(l\dot{\theta}(t)\right)^2 \\
&+\left(w_t(l,t)\right)^2+\left(w(x,t)\dot{\theta}\right)^2\} \\
&+1/2m_2\{2\dot{\eta}(t)\dot{\theta}(t)l\cos\left(\theta(t)\right) \\
&+2\dot{\eta}(t)w_t(l,t)\cos\left(\theta(t)\right) \\
&-2\dot{\eta}(t)\dot{\theta}(t)w\left(l,t\right)\sin\left(\theta(t)\right) \\
&+2w_t(l,t)\dot{\theta}(t)l\}
\end{aligned} \tag{4}
$$

The potential energy $E_p(t)$ due to the strain and gravity potential energy can be shown by:

$$E_p = E_{p1} + E_{p2} \tag{5}$$

In which

$$E_{p1} = 1/2\int_0^l EI\left(w_{xx}(x,t)\right)^2 dx \tag{6}$$

$$
\begin{aligned}
E_{p1} &= \rho g\int_0^l \begin{pmatrix} l - x\cos(\theta(t)) + \\ w(x,t)\sin(\theta(t))\end{pmatrix}dx \\
&+m_2 g\begin{pmatrix} l - l\cos(\theta(t)) \\ +w(l,t)\sin(\theta(t))\end{pmatrix}
\end{aligned} \tag{7}
$$

In order to obtain the equations of motion, the extended Hamiltonian principle is applied.

$$\int_{t_1}^{t_2}\left(\delta T - \delta V + \delta W_e\right)dt = 0 \tag{8}$$

δW_e, $\delta\eta$, δw and $\delta\theta$ represent the virtual work, the virtual base displacement, virtual elastic transverse displacement and virtual rigid body angular rotation respectively.

The total virtual work done on the system due to extrinsic disturbance and control forces on trolley and payload is given by:

$$
\begin{aligned}
\delta w_e &= (F_1(t))\delta\eta + (F_3(t)+d(t))\delta w(l,t) \\
&+(F_3(t)l + d(t)l + \tau(t))\delta\theta
\end{aligned} \tag{9}
$$

Assumption 1. For unknown boundary disturbance $d(t)$, its derivative $\dot{d}(t)$ is bounded by positive constant $D \in \mathbb{R}^+$, such that $\left|\dot{d}(t)\right| \le D$, $\forall(t)\in[0,\infty)$.

$F_1(t)$ and $F_3(t)$ are respectively the control forces on the trolley and the payload, and the control torque applied on the hub denoted as $\tau(t)$, $d(t)$ represents the extrinsic disturbance on the payload and by combining above equations and considering some simplification, equation (8) can be written as:

$$
\begin{aligned}
&\int_{t_2}^{t_1}\left(\sigma_1(x,t)\delta\theta\right)dt \\
&+\int_{t_2}^{t_1}\left(\sigma_2(x,t)\delta w\left(x,t\right)\right)dt \\
&+\int_{t_2}^{t_1}\left(\sigma_3(x,t)\delta r\right)dt \\
&+\int_{t_2}^{t_1}\left(\sigma_4(x,t)\delta w\left(l,t\right)\right)dt \\
&+\int_{t_2}^{t_1}\left(\sigma_5(x,t)\delta w\left(0,t\right)\right)dt = 0
\end{aligned} \tag{10}
$$

Letting $\sigma_1(x,t)=0$, $\sigma_2(x,t)=0$, $\sigma_3(x,t)=0$ and $\sigma_5(x,t)=0$, the equations of motion and the boundary conditions can be obtained as follows:

$$
\begin{aligned}
&\ddot{\theta}(t)\left(m_2 l^2+\rho l^3/3\right)+\\
&\left(\rho l^2/2+m_2 l\right)\ddot{\eta}(t)cos\left(\theta(t)\right)\\
&+\left(\rho l^2/2+m_2 l\right)gsin\left(\theta(t)\right)\\
&+\int_0^l \rho\begin{bmatrix}2w(x,t)\left(w_t(x,t)\right)\dot{\theta}(t)\\ +w^2(x,t)\ddot{\theta}(t)\\ +gw(x,t)cos\left(\theta(t)\right)\\ -\ddot{\eta}(t)w(x,t)sin\left(\theta(t)\right)\\ +w_{tt}(x,t)x\end{bmatrix}dx\\
&+m_2\begin{bmatrix}2w\left(l,t\right)\left(w_t\left(l,t\right)\right)\dot{\theta}(t)\\ +w\left(l,t\right)^2\ddot{\theta}(t)\\ +gw(l,t)cos\left(\theta(t)\right)\\ -\ddot{\eta}(t)w\left(l,t\right)sin\left(\theta(t)\right)\\ +w_{tt}(l,t)l\end{bmatrix}\\
&=\tau(t)+F_3(t)l+d(t)l
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\left(\rho l+m_1+m_2\right)\ddot{\eta}(t)\\
&+\left(\rho l^2/2+m_2 l\right)\ddot{\theta}(t)cos\left(\theta(t)\right)\\
&-\left(\rho l^2/2+m_2 l\right)\dot{\theta}^2(t)sin\left(\theta(t)\right)\\
&+\rho\int_0^l\begin{bmatrix}w_{tt}(x,t)cos\left(\theta(t)\right)\\ -2\left(w_t(x,t)\right)\dot{\theta}(t)sin\left(\theta(t)\right)\\ -\ddot{\theta}w(x,t)sin\left(\theta\right)\\ -\dot{\theta}^2(t)w(x,t)cos\left(\theta\right)\end{bmatrix}dx\\
&+m_2\begin{bmatrix}w_{tt}(l,t)cos\left(\theta(t)\right)\\ -2w_t(l,t)\dot{\theta}sin\left(\theta(t)\right)\\ -\ddot{\theta}(t)w\left(l,t\right)sin\left(\theta(t)\right)\\ -\dot{\theta}^2(t)w\left(l,t\right)cos\left(\theta(t)\right)\end{bmatrix}\\
&=F_1(t)
\end{aligned}
\tag{12}
$$

$$
\rho\begin{pmatrix}w_{tt}(x,t)+\ddot{\eta}(t)cos\left(\theta(t)\right)\\ -w(x,t)\dot{\theta}^2(t)\\ +gsin\left(\theta(t)\right)+\ddot{\theta}(t)x\end{pmatrix}+EIw_{xxxx}=0 \tag{13}
$$

$$
\begin{aligned}
&w(0,t)=w_x(0,t)=w_{xx}(l,t)=0\\
&m_2\begin{pmatrix}w_{tt}(l,t)+\ddot{\eta}(t)cos\left(\theta(t)\right)\\ -w(l,t)\dot{\theta}^2(t)+gsin\left(\theta(t)\right)\\ +l\ddot{\theta}(t)\end{pmatrix}-EIw_{xxxx}\\
&=F_3(t)+d(t)
\end{aligned}
\tag{14}
$$

3. Mathematical Preliminaries

For a better understanding of subsequent analysis, some lemmas are mentioned as following:

Lemma 1 [29]. Let $u_1(x,t), u_2(x,t) \in R$ with $x \in [0,L]$ and $t \in [0,\infty]$ the following inequalities hold:

$$
2u_1u_2 \le 2\left|u_1u_2\right| \le u^2{}_1+u^2{}_2 \ \forall u_1,u_2 \in R \tag{15}
$$

Similarly, from (1) we can show that:

$$
\left|u_1u_2\right|=\left|\left(\frac{1}{\sqrt{\delta}}u_1\right)\left(\sqrt{\delta}u_2\right)\right| \le \frac{1}{\delta}u^2{}_1+\delta u^2{}_2 \tag{16}
$$

Lemma 2 [29], [4] and [7]. Let $u(x,t) \in R$ with $x \in [0,L]$ and $t \in [0,\infty]$ which satisfies the boundary condition $u(0,t)=0$, $\forall$ the following inequalities hold: $t \in [0,\infty]$

$$
\frac{1}{l^2}\int_0^l u^2(x,t)dx \le \int_0^l u^2{}_x(x,t)dx \le l^2\int_0^l u^2{}_{xx}(x,t)dx \tag{17}
$$

$$
u^2(x,t) \le l\int_0^l u^2{}_x(x,t)dx \le l^3\int_0^l u^2{}_{xx}(x,t)dx \tag{18}
$$

In addition, if $u(x,t)$ satisfies the boundary condition $u_x(0,t)=0$, the following inequality holds:

$$
u_x{}^2(x,t) \le l\int_0^l u^2{}_{xx}(x,t)dx \tag{19}
$$

Equalities (15) , (16), (18) and (19) are used to express boundness of the Lyapunov function and its time derivative.

For expressing that along closed-loop process, all system signals remain bounded, equality (17) grounded in clamped boundary condition ($w(0,t)=0$), is utilized.

Assumption 2. Based on kinetic and potential energy in equations (4) and (6) for proposed crane system the following properties hold:

Property 1 [27], [28]. If the kinetic energy of system is bounded $\forall t \in [0,\infty)$ then $\left(\partial^n / \partial x^n\right) w_t(x,t)$ is bounded for $n=0,1,2$, $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$.

Property 2 [27], [28]. If the potential energy of system is bounded $\forall t \in [0,\infty)$ then $\left(\partial^n / \partial x^n\right) w_x(x,t)$ is bounded for $n=1,2$, $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$.

Remark 3. From a strictly mathematical point of view, one might question the above boundedness properties. However, from an engineering point of view, it appears rational to presume for a real physical system that if the energy of the system is bounded, then all the signals which make up the governing dynamic equations will also remain bounded [28].

4. Control design

Grounded in the control design, as can be seen in Figure 2, designing boundary signals F_1 , F_2 and $\tau(t)$, which are cooperatively applied on boundaries and hub, is the major accomplishment. To inspect the stability of the closed-loop system, signals that were mentioned above are utilized which help to reach the control laws: :(i) drive manipulator's base to the expected position η_d (ii) terminate the transverse vibration of the arm and control the rotational angle i.e., to guarantee $w(x,t)\rightarrow 0$ and $\theta(t)\rightarrow 0$ in the presence of the unknown varying disturbances. The following control laws are introduced:

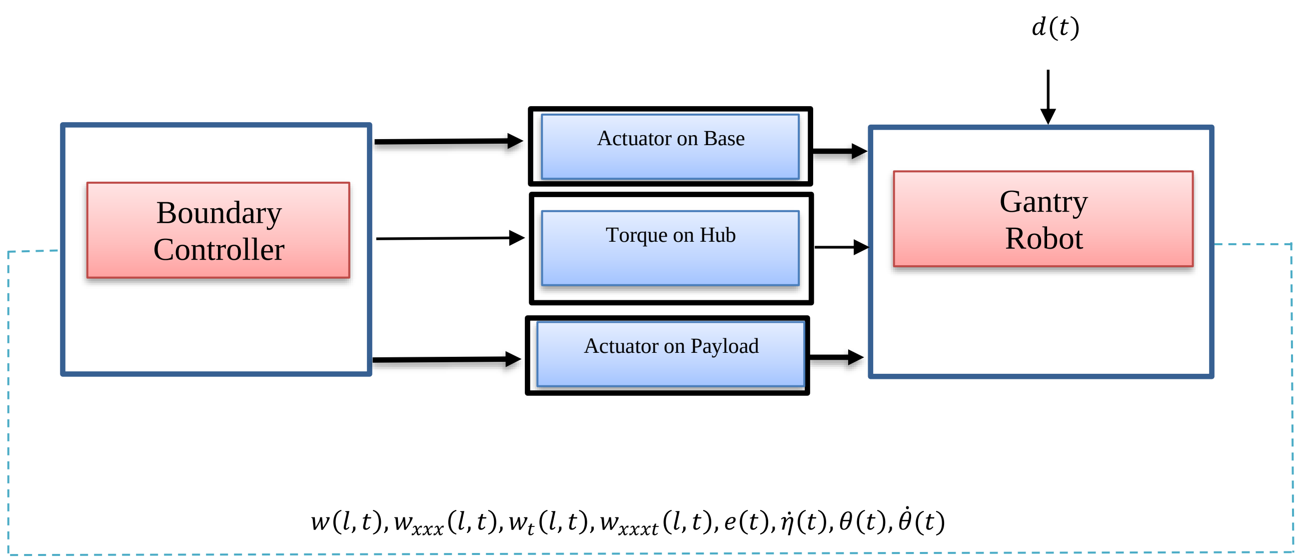

Figure 3 Block diagram of feedback control of flexible gantry manipulator system based on boundary control strategy

$$F_1(t) = -k_1\left(\varphi(t)\right) - m_2 w_{xxxt}(l,t) - EIw_{xxx}(l,t) + m_2 g\sin\left(\theta(t)\right) \tag{20}$$

$$\tau(t) = -k_3\dot{\theta}w^2(l,t) + \left(\rho l^2/2\right) g\sin\left(\theta(t)\right) + m_2 g\, w(l,t)\cos\left(\theta(t)\right) \tag{21}$$

$$F_3(t) = -k_2\dot{\eta}(t) \tag{22}$$

In which k_1, k_2, k_3 and k_4 are positive control gains and auxiliary function $\varphi(t)$ defines as:

$$\begin{aligned}\varphi(t) &= w_t(l,t) + l\dot{\theta}(t) \\ &- w_{xxx}(l,t) + k_s \sin^2(\theta(t))\end{aligned} \tag{23}$$

Remark 4. In implementing the control laws, some of the proposed feed backed signals can be measured by existing sensors and the rest can be obtained by calculations[13]. $w_{xxx}(l,t)$ can be measured by shear force sensor at boundary $w_t(l,t)$ and $w_{xxxt}(l,t)$ can be calculated by backward differencing. $\eta(t)$ and $\dot{\eta}(t)$ can be obtained by position and velocity sensors respectively. At last, $\theta(t)$ and its derivatives , $\dot{\theta}(t)$ can be calculated by rotary encoder and tachometer respectively. In this work, the Lyapunov candidate can be settled as:

$$\begin{aligned}V(t) &= V_1(t) + V_2(t) + V_3(t) \\ &+ \frac{k_p}{2}e(t)^2 + \frac{k_\theta}{2}sin^2(\theta(t))\end{aligned} \tag{24}$$

Where $\beta, \alpha, k_p, k_\theta$ are positive terms. The base position set point error $e(t)$ is definded in order to gain control objectives. V_1 and V_2 are explained grounded in the total mechanical energy of the string and the kinetic energy of the base and payload mass, respectively. in order to competence the stability procedure V_3 is explained. They are defined as:

$$\begin{aligned}V_1 &= 1/2\,\beta\rho\int_0^l \left\{ \begin{pmatrix} w_t(x,t) + x\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix}^2 + \begin{pmatrix} w(x,t)\dot{\theta}(t) \\ -\dot{\eta}(t)sin(\theta(t)) \end{pmatrix}^2 \right\} dx + \\ &1/2\,\beta\int_0^l EIw_{xx}^2(x,t)dx\end{aligned} \tag{25}$$

$$\begin{aligned}V_2 &= 1/2\,\beta m_2 \left\{ \begin{aligned}&(\phi(t) + \dot{\eta}(t)cos(\theta(t)))^2 \\ &+\beta \begin{pmatrix} w(l,t)\dot{\theta}(t) \\ -\dot{\eta}(t)sin(\theta(t)) \end{pmatrix}^2\end{aligned} \right\} \\ &+\beta 1/2\, m_1\dot{\eta}^2(t)\end{aligned} \tag{26}$$

$$\begin{aligned}V_3 &= \beta\rho g\int_0^l w(x,t)\sin(\theta(t))dx \\ &+\rho k_s \sin(\theta(t))\int_0^l \begin{pmatrix} w_t(x,t) \\ +x\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx \\ &+EI\,k_s \sin(\theta(t))w(l,t)\end{aligned} \tag{27}$$

$$V_4 = \alpha\rho\int_0^l xw_x(x,t) \begin{pmatrix} w_t(x,t) \\ +x\dot{\theta}(t) \\ +\dot{\eta}(t)cos(\theta(t)) \end{pmatrix} dx \tag{28}$$

Lemma3. The Lyapunov function candidate (24) is bounded by positive variables as following:

$$\begin{aligned}0 \le \lambda_{11} &\begin{pmatrix} \xi(t) + \int_0^l w_{xx}^2 dx + \phi^2(t) \\ +e^2(t) + sin^2(\theta(t)) + \dot{\eta}^2(t) \end{pmatrix} \le V \le \\ \lambda_{12} &\begin{pmatrix} \xi(t) + \int_0^l w_{xx}^2 dx + \phi^2(t) \\ +e^2(t) + sin^2(\theta(t)) + \dot{\eta}^2(t) \end{pmatrix}\end{aligned} \tag{29}$$

The auxiliary variable $\xi(t)$ is defined as:

$$\begin{aligned}\xi(t) &= \int_0^l w_t^2(x,t)dx \\ &+ \int_0^l w^2(x,t)\dot{\theta}^2(t)dx + l^2\dot{\theta}^2(t)\end{aligned} \tag{30}$$

Proof. Based on equations (16), (25) and adding two positive terms $\frac{k_p}{2}e^2, \frac{k_\theta}{2}sin^2(\theta)$ it is concluded that $V_1 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta)$ is upper bounded as:

$$0 \le V_1 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta) \le$$
$$2\beta\rho\int_0^l w_t^{\,2}(x,t)dx$$
$$+1/2\,\beta\int_0^l EIw_{xx}^{\,2}(x,t)dx + 2\beta\rho l\,\dot{\eta}^2cos^2(\theta) \quad (31)$$
$$\beta\rho\int_0^l w^2(x,t)\dot{\theta}^2(t)dx + \beta\rho l\,\dot{\eta}^2 sin^2(\theta)$$
$$+2\beta\rho l^3\dot{\theta}^2 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta)$$

By having $\beta\rho l\dot{\eta}^2 \le 2\beta\rho l\dot{\eta}^2cos^2(\theta)$ $+\beta\rho l\dot{\eta}^2 sin^2(\theta) \le 2l\,\beta\rho\dot{\eta}^2$ and based on definition of $\xi(t)$ in equation (30) we can rewrite equality (31) as:

$$\min(\beta\rho,\beta\rho l)\xi(t) + \beta/2\int_0^l EIw_{xx}^{\,2}(x,t)dx$$
$$+\beta\rho l\dot{\eta}^2 + \frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta) \le V_1$$
$$\le \max(2\beta\rho,2\beta\rho l)\xi(t) \quad (32)$$
$$+\beta/2\int_0^l EIw_{xx}^{\,2}(x,t)dx + 2\beta\rho l\dot{\eta}^2$$
$$+\frac{k_p}{2}e^2 + \frac{k_\theta}{2}sin^2(\theta)$$

From definition of V_2 in equation (10) and inequality (1) and by having $b_1 = 2\beta\max\left(\frac{m_{1,}}{2}m_2\right)$ and $b_2 = 2\beta\min\left(\frac{m_{1,}}{2}m_2\right)$ which is positive weighting constant, we can show that is bounded as following:

$$b_1\left[\phi^2(t) + \dot{\eta}^2(t) + w^2(l,t)\dot{\theta}^2(t)\right] \le V_2$$
$$\le b_2\left[\phi^2(t) + \dot{\eta}^2(t) + w^2(l,t)\dot{\theta}^2(t)\right] \quad (33)$$

Also in similar manner based on equations (15) , (16) and (27) it is shown that:

$$|V_3 + V_4| \le 2\alpha\rho l\,\xi(t)$$
$$+\alpha\rho l\int_0^l w_{xx}^{\,2}(x,t)dx$$
$$+2\alpha\rho l\dot{\eta}^2cos^2(\theta) \quad (34)$$
$$+\beta\rho g\,\sin^2(\theta(t))$$
$$+\beta\rho g\int_0^l w^2(x,t)dx$$

Based on relations (17) and (18) it can be shown $\int_0^l w^2(x,t)dx \le l^4\int_0^l w^2_{xx}(x,t)dx$ and $w^2(l,t) \le l^3\int_0^l w^2_{xx}(x,t)dx$ respectively. So relation (34) can be rewritten as:

$$|V_3 + V_4| \le$$
$$4 + \left(k_s\left(\rho + EIl^3\right) + \beta\rho g l^2 + \alpha\rho l\right)$$
$$\times\int_0^l w^2_{xx}(x,t)dx\,\left(\alpha\rho l + \rho k_s\right)\xi(t) \quad (35)$$
$$+4\rho l k_s\dot{\eta}^2 +$$
$$\left(\rho k_s + \beta\rho g + EI\right)\sin^2(\theta(t))$$

By adding equalities (31) ,(33) and (35) based on defining positive constants as following:

$$\lambda_1 = \min(\beta\rho,\beta\rho l) - 4(\alpha\rho l + \rho k_s) > 0$$
$$\lambda_2 = 2\max(\beta\rho,\beta\rho l) + 4(\alpha\rho l + \rho k_s)$$
$$\lambda_3 = \beta/2\,EI - \left(k_s\left(\rho + EIl^3\right) + \beta\rho g l^2 + \alpha\rho l\right) > 0$$
$$\lambda_4 = \beta/2\,EI + \left(k_s\left(\rho + EIl^3\right) + \beta\rho g l^2 + \alpha\rho l\right)$$

Furthermore $\lambda_5 = b_1 + 2\beta\rho l - 4l(\alpha\rho + \rho k_s) > 0$ and $\lambda_6 = b_1 + \beta\rho + 4l(\alpha\rho + \rho k_s)$. Considering, $\lambda_7 = \frac{k_\theta}{2} - (\rho k_s + \beta\rho g + EI) > 0$ and $\lambda_8 = \frac{k_\theta}{2} + (EI + \rho k_s + \beta\rho g)$, the Lyapunov candidate function relation (29) is proved. where $\lambda_9 = \min(\lambda_1,\lambda_3,\lambda_5,\lambda_7)$ and $\lambda_{10} = \max(\lambda_2,\lambda_4,\lambda_6,\lambda_8)$ are positive constants.

Theorem 1. For the system dynamics described by (11) , (12) and (13) , boundary conditions (14), using the proposed boundary control laws (20), (21) and (22) , if the initial conditions are bounded, then the system is regulated asymptotically in the following sense:

$\lim_{t\to\infty}|w(x,t)|=0$, $\lim_{t\to\infty}|\theta(t)|=0$, $\lim_{t\to\infty}|e(t)|=0$

Proof. Differentiating Lyapunov candidate equation (24) respect to time leads to:

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t)\\ &+\dot{V}_4(t)+k_\theta\dot{\theta}(t)sin(\theta)\end{aligned} \tag{36}$$

The first term of equation (36) can be represented as:

$$\dot{V}_1(t)=E_1(t)+E_2(t) \tag{37}$$

In which $E_1(t),E_2(t)$ can be shown as:

$$\begin{aligned}E_1 &= \beta\rho\int_0^l\left\{\begin{matrix}(w_t)(w_{tt})+(w_t)w\,\dot{\theta}^2\\ +\ddot{\eta}(w_t)cos(\theta)+(w_t)\ddot{\theta}x\end{matrix}\right\}dx\\ &+\beta\int_0^l EIw_{xxt}w_{xx}dx\end{aligned} \tag{38}$$

$$\begin{aligned}E_2 = \beta\rho\int_0^l &\{\dot{\eta}\ddot{\eta}+x\,\dot{\theta}\ddot{\theta}+w^2\dot{\theta}\ddot{\theta}\\ &+\ddot{\eta}\dot{\theta}xcos(\theta)+\dot{\eta}\ddot{\theta}xcos(\theta)\\ &-\dot{\eta}\dot{\theta}^2xsin(\theta)+\\ &\dot{\eta}(w_{tt})cos(\theta)-\dot{\eta}(w_t)\dot{\theta}sin(\theta)\\ &-\ddot{\eta}\dot{\theta}wsin(\theta)-\dot{\eta}\ddot{\theta}wsin(\theta)\\ &-\dot{\eta}\dot{\theta}(w_t)sin(\theta)\\ &-\dot{\eta}\dot{\theta}^2wcos(\theta)+(w_{tt})\dot{\theta}x\,\}dx\end{aligned} \tag{39}$$

By substituting equation of motion (13) into equation (38) we obtain:

$$\begin{aligned}E_1(t) &= -\beta\int_0^l EIw_t(x,t)w_{xxxx}(x,t)dx\\ &+\beta\int_0^l EI\,w_{xxt}w_{xx}dx\\ &+2\beta\rho\int_0^l w_t\,w\,\dot{\theta}^2dx\\ &-\beta\rho\int_0^l g\,(w_t)sin(\theta(t))dx\end{aligned} \tag{40}$$

By integration by part and from the definition of auxiliary function $\varphi(t)$, equation (40) yields to:

$$\begin{aligned}E_1 &\le \beta EI/2\varphi(t)^2\\ &-\beta EI\begin{pmatrix}w^2_{xxx}(l,t)+w^2_t(l,t)\\ +l^2\dot{\theta}^2+k_s^2\sin^2(\theta(t))\end{pmatrix}\\ &-\beta EI\begin{pmatrix}w_t(l,t)+k_s\sin(\theta(t))\\ +w_{xxx}(l,t)\end{pmatrix}l\dot{\theta}\\ &-EIk_s\sin(\theta(t))\begin{pmatrix}-w_{xxx}(l,t)\\ +w_t(l,t)\end{pmatrix}\\ &-\beta\rho\int_0^l g\,(w_t)sin(\theta(t))dx\end{aligned} \tag{41}$$

Based on (16), differentiating $V_2(t)$ yields to:

$$\dot{V}_2 = \beta m_2 \left[\phi(t) \begin{pmatrix} w_{tt}(l,t) \\ -\dot{\eta}(t)\dot{\theta}(t) sin\left(\theta(t)\right) \\ +l\ddot{\theta}(t) \\ +\ddot{\eta}(t) cos\left(\theta(t)\right) \end{pmatrix} \right]$$
$$+\beta m_2 \left[\begin{array}{l} w_{tt}(l,t) \begin{pmatrix} l\dot{\theta}(t) \\ +\dot{\eta}(t) cos\left(\theta(t)\right) \end{pmatrix} \\ +l\ddot{\theta}(t)\dot{\eta}(t) cos\left(\theta(t)\right) \end{array} \right]$$
$$+\beta m_2 \left[l\dot{\theta}(t) \begin{pmatrix} l\ddot{\theta}(t) + \ddot{\eta} cos\left(\theta(t)\right) \\ -l\dot{\theta}(t)\dot{\eta}(t) sin\left(\theta(t)\right) \end{pmatrix} \right] \tag{42}$$
$$+\beta m_2 \begin{bmatrix} w\left(w_t\right)\dot{\theta}^2 \\ +w^2\ddot{\theta}\dot{\theta} - \ddot{\eta} w\dot{\theta} sin\left(\theta\right) \\ -\dot{\eta} w_t \dot{\theta} sin\left(\theta\right) \\ -\dot{\eta} w\ddot{\theta} sin\left(\theta\right) - \dot{\eta} w\dot{\theta}^2 cos\left(\theta\right) \end{bmatrix}$$
$$+\beta(m_1+m_2)\dot{\eta}(t)\ddot{\eta}(t)$$
$$+\beta m_2 w_{xxxt}(l,t)\phi^2(t)$$

Differentiating $V_3(t)$ yields to:

$$\dot{V}_3(t) = \left(k_s \dot{\theta} cos\left(\theta\right)\right) \int_0^l \begin{pmatrix} w_t(x,t) + x\dot{\theta}(t) \\ +\dot{\eta}(t) cos\left(\theta(t)\right) \end{pmatrix} dx$$
$$+\left(k_s \sin(\theta)\right) \int_0^l \begin{pmatrix} w_{tt}(x,t) + x\ddot{\theta}(t) \\ +\ddot{\eta}(t) cos\left(\theta(t)\right) \\ -\dot{\eta}(t)\dot{\theta}(t) sin\left(\theta(t)\right) \end{pmatrix} dx$$
$$+\beta\rho g \int_0^l w_t(x,t)\sin(\theta(t))dx \tag{43}$$
$$+\beta\rho g \int_0^l w(x,t)\dot{\theta}(t)\cos(\theta(t))dx$$
$$-E\,I\,k_s \dot{\theta} cos\left(\theta\right) w(l,t)$$
$$+E\,I\,k_s \sin(\theta(t)) w_t(l,t)$$

By substituting equation of motion (13) into equation (43) we obtain:

$$\dot{V}_3(t) = k_s \sin(\theta) \int_0^l \begin{bmatrix} -E\,I\,w_{xxxx}(x,t) \\ +\rho w\dot{\theta}^2 \\ -\rho g sin\left(\theta\right) \\ -\rho\dot{\eta}\dot{\theta} sin\left(\theta\right) \end{bmatrix} dx$$
$$+k_s \dot{\theta} cos\left(\theta\right) \int_0^l \begin{pmatrix} w_t(x,t) + x\dot{\theta}(t) \\ +\dot{\eta}(t) cos\left(\theta(t)\right) \end{pmatrix} dx$$
$$+\beta\rho g \int_0^l w_t(x,t)\sin(\theta(t))dx$$
$$+\beta\rho g\dot{\theta}(t) cos\left(\theta(t)\right) \int_0^l w(x,t)dx \tag{44}$$
$$+\beta\rho g \int_0^l w_t(x,t)\sin(\theta(t))dx$$
$$+\beta\rho g \int_0^l w(x,t)\dot{\theta}(t)\cos(\theta(t))dx$$
$$-EIk_s\dot{\theta}(t)\cos(\theta(t))w(l,t)$$
$$+\beta EIk_s \sin(\theta) w_t(l,t)$$

Furthermore based on $0 \le cos\left(\theta\right) \le 1$ we have:

$$k_s \sin(\theta) \int_0^l \begin{bmatrix} -EIw_{xxxx}(x,t) \\ +\rho w\dot{\theta}^2 \\ -\rho g sin\left(\theta\right) \\ -\rho\dot{\eta}\dot{\theta} sin\left(\theta\right) \end{bmatrix} dx \le$$
$$\frac{\rho k_s}{\delta_1}\dot{\theta}^2 + k_s\rho\delta_1 \int_0^l w^2\dot{\theta}^2 dx \tag{45}$$
$$-k_s\rho gl \sin^2(\theta)$$
$$-k_s\rho l\dot{\eta}\dot{\theta} sin^2\left(\theta\right)$$
$$-k_s EIw_{xxx}(l)\sin(\theta)$$

$$k_s\rho\dot{\theta} cos\left(\theta\right) \int_0^l \begin{pmatrix} w_t(x,t) + x\dot{\theta}(t) \\ +\dot{\eta}(t) cos\left(\theta(t)\right) \end{pmatrix} dx \le$$
$$k_s\rho\delta_2\dot{\theta}^2 + \frac{\rho k_s}{\delta_2}\int_0^l w_t^2(x,t)dx \tag{46}$$
$$+k_s\rho \, l^2/2 \, \dot{\theta}^2 + k_s\rho l\,\dot{\eta}(t)\dot{\theta} cos^2\left(\theta\right)$$

$$\beta\rho g\dot{\theta}(t) cos\left(\theta(t)\right) \int_0^l w(x,t)dx \le$$
$$\beta\rho g\,\delta_3\dot{\theta}^2(t) + \frac{\beta\rho g l^4}{\delta_3}\int_0^l w_{xx}^2(x,t)dx \tag{47}$$

In similar way by differentiating $V_4(t)$ one can attain:

$$\dot{V}_4 = \alpha\rho\int_0^l x w_{xt}(x,t)\begin{pmatrix} w_t(x,t)+ \\ x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right) \end{pmatrix}dx + \alpha\rho\int_0^l x w_x(x,t)\begin{pmatrix} w_{tt}(x,t) \\ +x\,\ddot{\theta}(t) \\ +\ddot{\eta}(t)cos\left(\theta(t)\right) \\ -\dot{\eta}(t)\dot{\theta}(t)sin\left(\theta(t)\right) \end{pmatrix}dx \quad (48)$$

By substituting equation of motion (13) into (48) and some simplification yields to:

$$\dot{V}_4 = \alpha\int_0^l \rho x\, w_{xt}(x,t)\begin{pmatrix} w_t(x,t) \\ +x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right) \end{pmatrix}dx - \alpha EI\int_0^l x w_x(x,t)\, w_{xxxx}(x,t)dx + \alpha\int_0^l \rho x w_x\begin{pmatrix} w\,\dot{\theta}^2 + gsin\left(\theta\right) \\ -\dot{\eta}\,\dot{\theta}sin\left(\theta\right) \end{pmatrix}dx \quad (49)$$

Equation (49) can represented as:

$$\dot{V}_4(t) = A_1(t) + A_2(t) + A_3(t) \quad (50)$$

which:

$$A_1(t) = \alpha\int_0^l \rho x w_{xt}(x,t)\begin{pmatrix} w_t(x,t) \\ +x\,\dot{\theta}(t) \\ +\dot{\eta}(t)cos\left(\theta(t)\right) \end{pmatrix}dx \quad (51)$$

$$A_2(t) = \alpha\int_0^l \rho x w_x\begin{pmatrix} w\,\dot{\theta}^2 \\ +gsin\left(\theta\right) \\ -\dot{\eta}\dot{\theta}sin\left(\theta\right) \end{pmatrix}dx \quad (52)$$

$$A_3(t) = -\alpha EI\int_0^l \{x\, w_x(x,t)\, w_{xxxx}(x,t)\}dx \quad (53)$$

Integrating equation (51) leads to:

$$A_1(t) = \alpha\rho l\,\frac{w_t^{\,2}\left(l,t\right)}{2} - \alpha\rho l\int_0^l \frac{w_t^{\,2}}{2}dx + \alpha\rho l^2\dot{\theta}w_t - 2\alpha\rho l\int_0^l x\,\dot{\theta}w_t dx + \alpha\int_0^l \rho x w_{xt}\left(\dot{r}cos\left(\theta\right)\right)dx \quad (54)$$

Based on relation (16) it can be shown that:

$$\alpha\int_0^l \rho x w_{xt}(x,t)\left(\dot{\eta}(t)cos\left(\theta(t)\right)\right)dx < \frac{\alpha\rho l}{\sigma_1}w_t^{\,2}(l,t) + \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}}\int_0^l w_t(x,t)^2 dx \left(\alpha\rho l\,\sigma_1 + \frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2\left(\theta(t)\right) \quad (55)$$

$$\alpha\rho l^2\dot{\theta}w_t(l,t) + 2\alpha\rho\int_0^l x\,\dot{\theta}w_t(x,t)dx \le \left(\frac{\alpha\rho l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\dot{\theta}^2(t) \times\left(2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\int_0^l w_t^{\,2}(x,t)dx + \alpha\rho l^2\sigma_3 w_t^{\,2}(l,t) \quad (56)$$

By substituting relations (55) and (56) into (54) it can be obtained:

$$A_1(t) \le -\left(\frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}}\right)\int_0^l w_t^{\,2}(x,t)dx + \left(\frac{\alpha\rho l}{2} + \frac{\alpha\rho l}{\sigma_1} + \alpha\rho l^2\sigma_3\right)w_t^{\,2}\left(l,t\right) \left(\frac{\alpha\rho l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\dot{\theta}^2(t) + \left(\alpha\rho l\,\sigma_1 + \frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2\left(\theta(t)\right) \quad (57)$$

Integrating equation (52) leads to:

$$A_2(t) = \frac{\alpha\rho l}{2}\dot{\theta}^2 w^2(l,t) - \alpha\rho\dot{\theta}^2\int_0^l w^2 dx + \alpha\int_0^l \rho x\, w_x\left(g\sin(\theta) - \dot{\eta}\dot{\theta}\sin(\theta)\right)dx \tag{58}$$

By integrating the third part of equation (58) on can obtain:

$$\begin{aligned}\alpha\int_0^l \rho x w_x\left(g\sin(\theta) - \dot{\eta}\dot{\theta}\sin(\theta)\right)dx \le\\ -\alpha\rho l\,\dot{\eta}(t)\dot{\theta}(t)w(l,t)\sin(\theta)(t)\\ +\alpha\rho l\,\dot{\eta}(t)\dot{\theta}(t)\sin(\theta(t))\int_0^l w(x,t)dx\\ \frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}\sin^2(\theta)\\ +\alpha\rho g\,\sigma_5\sqrt{\frac{l^3}{3}}\int_0^l w_x(x,t)dx\end{aligned} \tag{59}$$

Relation (59) also can be rewritten as:

$$\begin{aligned}\alpha\int_0^l \rho x w_x\left(g\sin(\theta) - \dot{\eta}\dot{\theta}\sin(\theta)\right)dx \le\\ \left(\alpha\rho l\,\sigma_6 + \frac{\alpha\rho l}{\sigma_7}\right)\dot{\eta}^2(t)\sin^2(\theta(t))\\ +\alpha\rho l\,\sigma_7\dot{\theta}^2(t)\int_0^l w^2(x,t)dx\\ +\frac{\alpha\rho l}{\sigma_6}\dot{\theta}^2(t)w^2(l,t)\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}\sin^2(\theta)\\ +\alpha\rho g\,\sigma_5 l^2\sqrt{\frac{l^3}{3}}\int_0^l w_{xx}(x,t)dx\end{aligned} \tag{60}$$

Based on inequality (60), equation (58) yields to:

$$\begin{aligned}A_2(t) \le -\left(\alpha\rho - \alpha\rho l\,\sigma_7\right)\dot{\theta}^2\int_0^l w^2 dx\\ +\left(\alpha\rho\frac{l}{2} + \frac{\alpha\rho l}{\sigma_6}\right)\dot{\theta}^2(t)w^2(l,t)\\ \left(\alpha\rho l\,\sigma_6 + \frac{\alpha\rho l}{\sigma_7}\right)\dot{\eta}^2(t)\sin^2(\theta(t))\\ +\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}\sin^2(\theta)\\ +\alpha\rho g\,\sigma_5 l^2\sqrt{\frac{l^3}{3}}\int_0^l w_{xx}(x,t)dx\end{aligned} \tag{61}$$

By integrating by parts, $A_3(t)$ yields to:

$$\begin{aligned}A_3(t) = -\alpha EIL w_x(l,t)w_{xxx}(l,t)\\ +\alpha EI\int_0^L w_x(x,t)w_{xxx}(x,t)dx\\ +\alpha EI\int_0^L x\, w_{xx}(x,t)w_{xxx}(x,t)dx\end{aligned} \tag{62}$$

By integrating relation(62), $A_2(t)$ can be obtained as follows:

$$\begin{aligned}A_3 \le -\alpha EIL w_x(l,t)w_{xxx}(l,t)\\ -3\alpha EI/2\int_0^L w^2_{xx}(x,t)dx\end{aligned} \tag{63}$$

Based on (16) and (19) Relation (63) also can be rewritten as:

$$\begin{aligned}A_3 \le \frac{\alpha EIl}{\sigma_8}w^2_{xxx}(l,t)\\ -\alpha EI\left(3/2 - l\,\sigma_8\right)\int_0^L w^2_{xx}(x,t)dx\end{aligned} \tag{64}$$

By substituting (57), (61) and (64) into (50), $\dot{V}_4$ can be determined as following:

$$\begin{aligned}
\dot{V}_4 \le & -\alpha\left(EI\left(3/2 - l\sigma_8\right) - \rho g\sigma_5 l^2\sqrt{\frac{l^3}{3}}\right) \\
& \int_0^l w_{xx}^{\ 2}(x,t)dx \\
& -\left(\alpha\rho - \alpha\rho l\sigma_7\right)\dot{\theta}^2\int_0^l w^{\ 2}dx \\
& -\left(\frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}}\right) \\
& \int_0^l w_t^{\ 2}(x,t)dx \\
& +\left(\frac{\alpha\rho l}{2} + \frac{\alpha\rho l}{\sigma_1} + \alpha\rho l^2\sigma_3\right) \\
& w_t^{\ 2}(l,t)\frac{\alpha EIl}{\sigma_8}w_{xxx}^{\ 2}(l,t) \\
& +\left(\frac{\alpha\rho l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\alpha\rho\sigma_4\right)\dot{\theta}^2(t) \\
& +\left(\alpha\rho\frac{l}{2} + \frac{\alpha\rho l}{\sigma_6}\right)\dot{\theta}^2(t)w^{\ 2}(l,t) \\
& +\left(\alpha\rho l\sigma_1 + \frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2(\theta(t)) \\
& +\left(\alpha\rho l\sigma_6 + \frac{\alpha\rho l}{\sigma_7}\right)\dot{\eta}^2(t)sin^2(\theta(t)) \\
& +\frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}sin^2(\theta)
\end{aligned} \tag{65}$$

By introducing following relations as follows:

$$\begin{aligned}
& \left(\alpha\rho l\sigma_1 + \frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}\right)\dot{\eta}^2(t)cos^2(\theta(t)) \\
& +\left(\alpha\rho l\sigma_6 + \frac{\alpha\rho l^2}{\sigma_7}\right)\dot{\eta}^2(t)sin^2(\theta(t)) \\
& \le \lambda\left(\dot{\eta}^2 sin^2(\theta) + \dot{\eta}^2 cos^2(\theta)\right) \le \lambda\dot{\eta}^2
\end{aligned} \tag{66}$$

In which

$$\lambda = \max\left(\alpha\rho l\sigma_1 + \frac{\alpha\rho}{\sigma_2}\sqrt{\frac{l^3}{3}}, \alpha\rho l\sigma_6 + \frac{\alpha\rho l^2}{\sigma_7}\right)$$

and also:

By substituting equations (64), (65) and (66) into (36), then we have:

$$\begin{aligned}
\dot{V} \le & \\
& -\begin{pmatrix} \alpha EI\left(3/2 - l\sigma_8\right) \\ -\alpha\rho g\sigma_5 l^2\sqrt{\frac{l^3}{3}} - \frac{\beta\rho g l^4}{\delta_3}\end{pmatrix}\int_0^l w_{xx}^{\ 2}(x,t)dx \\
& -\left(\alpha\rho - \alpha\rho l\sigma_7 - k_s\rho\delta_1\right)\dot{\theta}^2(t) \\
& \times\int_0^l w^{\ 2}(x,t)dx \\
& -\left(\frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}} - \frac{\rho k_s}{\delta_2}\right) \\
& \times\int_0^l w_t^{\ 2}(x,t)dx \\
& -\left(\beta k_1 - \beta EI/2\right)\varphi(t)^2 \\
& -\begin{pmatrix} \beta k_2 + \frac{\beta EIl^2}{2} - \alpha\rho\left(\frac{l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\sigma_4\right) \\ +\frac{\rho k_s}{\delta_1} - k_s\rho\delta_2 - k_s\rho\, l^2/2 - \beta\rho g\,\delta_3\end{pmatrix}\dot{\theta}^2(t) \\
& -\left(\beta k_3 - \alpha\rho\frac{l}{2} - \frac{\alpha\rho l}{\sigma_6}\right)\dot{\theta}^2(t)w^{\ 2}(l,t) \\
& -\left(\frac{\beta EI}{2} - \frac{\alpha EIl}{\sigma_8}\right)w_{xxx}^{\ 2}(l,t) - \left(\beta k_4 - \lambda\right)\dot{\eta}^2 \\
& -\left(\frac{\beta EI}{2} - \frac{\alpha\rho l}{2} - \frac{\alpha\rho l}{\sigma_1} - \alpha\rho l^2\sigma_3\right)w_t^{\ 2}(l,t) \\
& -\left(k_s^{\ 2} + k_s\rho g l - \frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}}\right)\sin^2(\theta)
\end{aligned} \tag{67}$$

The positive parameters $k_1 - k_4$, k_p, k_θ, k_{e1}, k_s, δ_{1-11}, σ_{1-8}, α, and β are determined in order to fulfill following relations:

$$\begin{aligned}
\upsilon_1 = & \alpha EI\left(3/2 - l\sigma_8\right) \\
& -\alpha\rho g\sigma_5 l^2\sqrt{\frac{l^3}{3}} - \frac{\beta\rho g l^4}{\delta_3} > 0
\end{aligned} \tag{68}$$

$$\upsilon_2 = \alpha\rho - \alpha\rho l\sigma_7 - k_s\rho\delta_1 > 0 \quad (69)$$

$$\upsilon_3 = \frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}} - \frac{\rho k_s}{\delta_2} > 0 \quad (70)$$

$$\upsilon_4 = \frac{\alpha\rho l}{2} - 2\sqrt{\frac{l^3}{3}}\frac{\alpha\rho}{\sigma_4} - \alpha\rho\sigma_2\sqrt{\frac{l^3}{3}} - \frac{\rho k_{e1}}{\delta_4} - \frac{\rho k_s}{\delta_9} > 0 \quad (71)$$

$$\upsilon_5 = \beta k_1 - \beta EI/2 > 0 \quad (72)$$

$$\upsilon_6 = \beta k_2 + \frac{\beta EI l^2}{2} - \alpha\rho\left(\frac{l^2}{\sigma_3} + 2\sqrt{\frac{l^3}{3}}\sigma_4\right) + \frac{\rho k_s}{\delta_1} - k_s\rho\delta_2 - k_s\rho l^2/2 - \beta\rho g\,\delta_3 > 0 \quad (73)$$

$$\upsilon_7 = \beta k_3 - \alpha\rho\frac{l}{2} - \frac{\alpha\rho l}{\sigma_6} > 0 \quad (74)$$

$$\upsilon_8 = \frac{\beta EI}{2} - \frac{\alpha EIl}{\sigma_8} > 0 \quad (75)$$

$$\upsilon_9 = \beta k_4 - \lambda > 0 \quad (76)$$

$$\upsilon_{10} = \frac{\beta EI}{2} - \frac{\alpha\rho l}{2} - \frac{\alpha\rho l}{\sigma_1} - \alpha\rho l^2\sigma > 0 \quad (77)$$

$$\upsilon_{11} = \frac{\beta EI}{2} - \frac{\alpha\rho l}{2} - \frac{\alpha\rho l}{\sigma_1} - \alpha\rho l^2\sigma_3 > 0 \quad (78)$$

$$\upsilon_{12} = k_s^2 + k_s\rho g l - \frac{\alpha\rho g}{\sigma_5}\sqrt{\frac{l^3}{3}} > 0 \quad (79)$$

Also relation (67) implies that $\dot{V} \le -\lambda_{11}\left(\begin{array}{l}\xi(t)\\ +\int_0^l w_{xx}^2(x,t)dx\\ +\phi^2(t) + \dot{\eta}(t)^2 + \sin^2(\theta)\end{array}\right)$, in which λ_{13} is positive constant by definition $\lambda_{13} = \min(\upsilon_1, \upsilon_2, \upsilon_3, ..., \upsilon_{12})$. Also there exist

$$\begin{array}{l}\xi(t) + \int_0^l w_{xx}^2(x,t)dx + \phi^2(t)\\ +e^2 + \dot{\eta}(t)^2 + \sin^2(\theta)\end{array}$$ is bounded.

Since all terms are all positive, then $\xi(t), w_{xx}^2(x,t), \phi^2(t)$ and $\dot{\eta}(t)^2$ are all bounded, based on boundedness of $\xi(t)$, $w_t(x,t)$, $\dot{\theta}(t)$ are bounded $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$. So boundedness of the total mechanical energy of the string system is gained. From properties 1 and 2, $w_{xxxt}(x,t)$ and $w_{xxxx}(x,t)$ are bounded $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$. From equation of motions (11), (12) and (13) it will be concluded that $w_{tt}(x,t)$, $\ddot{\eta}(t)$ and $\ddot{\theta}(t)$ are all bounded $\forall t \in [0,\infty)$ and $\forall x \in [0,l]$. Grounded in all above statements it can be concluded that the control signals are bounded and all the signals in closed loop system remains bounded.

5. Simulation Results

The solution for the suggested system that has been expressed by equations (11), (12) and (13) with the boundary conditions (14), grounded in finite difference method is reached and for solving the differential equations [11] , the above mentioned solution dispenses an exact process. Proposed system which is excited by boundary disturbance $d(t) = 0.1\sin(0.1\pi t)$ with initial conditions $\theta_0 = 40^0$, $\eta_0 = 0$, $w(x,0) = 0$ and $w_t(x,0) = 0$ is considered. Demonstrating the performance of the control rules (20), (21) , (22) and disturbance observer is the main purpose of this section. The other specifications are introduced in Table 1.

Table 1. Gantry manipulator specifications

Parameter	Description	Value
l	Length of manipulator	$0.8\ m$
m_1	Mass of base	$5\ kg$
m_2	Mass of tip payload	$4\ kg$
ρ	Uniform mass per unit length	$0.2\ kg/m$
EI	Bending stiffness of manipulator	$8\ Nm^2$
I	Inertial of the hub	$0.3\ kg/m^2$
η_d	Desired position	$1\ m$

In order to evaluate the control performance, two cases are compared. First, the system behavior lacking control inputs is demonstrated. Then by exerting control inputs on the system, the impacts of the control inputs are represented. In Figures 4-7, The manipulator position, the three-dimensional delineation of the transverse vibration of the manipulator, the deflection of the four equi-distanced points on the link, and the angular position of the proposed system which all of them are without control, are illustrated, respectively. This show that the system is thoroughly unstable in the absence of feedback control inputs.

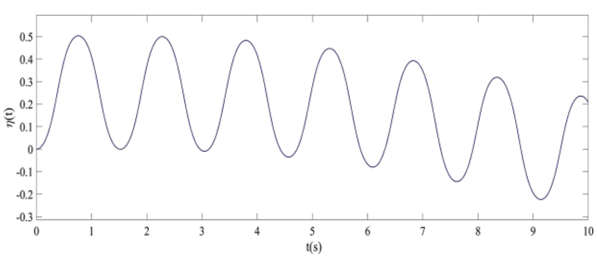

Figure 4. Manipulator position without control

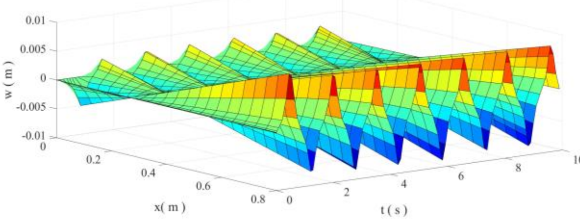

Figure 5. Transverse displacement of flexible manipulator without control

As can be seen, because of the boundary disturbances, the manipulator and the angular position are far beyond from the desired position and regulation point, respectively. Furthermore, the proposed system is affected by the large vibrations that depend on the manipulator's length which cannot be neglected.

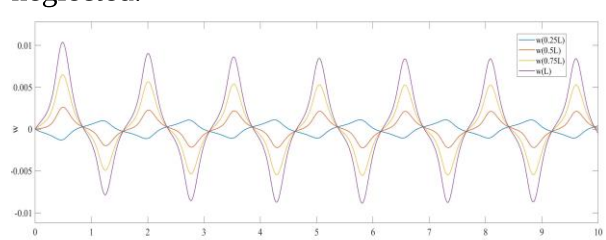

Figure 6. The deflection of the four equi-distanced points on the link without control

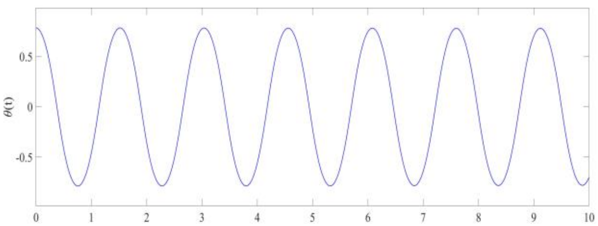

Figure 7. The angular position of flexible manipulator without control

In the second case, to gain control objectives and guarantee the best performance of the controller, the simulation performed by exerting introduced control rules with appropriate parameters. Mentioned control gains are taken cautiously by a large number of trials. Furthermore, they must satisfy relations $\delta_{1-11}, \sigma_{1-8}$ and (29) which all lead to reaching the best performance of the controller.

Accordingly, the proposed parameters are chosen as $\beta = 3$, $\alpha = 0.01$, $k_{e1} = 0.02$, $k_s = 0.01$, $\delta_{1-2} = 10$, $\delta_3 = 1$, $\delta_4 = 0.52$, $\delta_5 = 98$, $\delta_{6-9} = 1$, $\delta_{10} = 0.1$, $\delta_{11} = 1$, $\delta_{12} = 26$, $\delta_{13} = 12.4$, $\delta_{14-15} = 1$, $k_1 = 1100$, $k_2 = 7$, $k_3 = 1.5$, $k_4 = 1$, $k_\theta = 70$ and $k_p = 260$.

In Figure 8, it can be seen that the system is driven to the demanded position within 6 seconds. It is noteworthy that because of the feedback control inputs, the deflections as a consequence of vibration are suppressed. In Figures 9-10 , the feedback proposed control inputs is shown. Furthermore, based on Figure 11, the angular position is greatly regulated within 7 Sec.

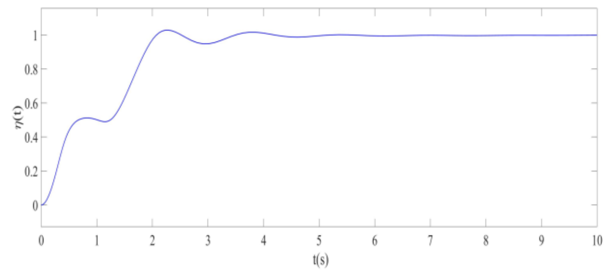

Figure 8. Manipulator position with control

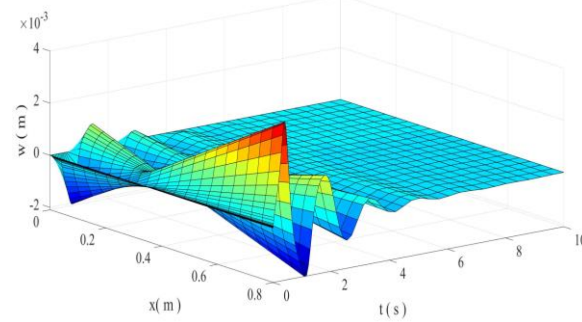

Figure 9. Transverse displacement $w\ (x\ ,t\ )$ of flexible manipulator without control

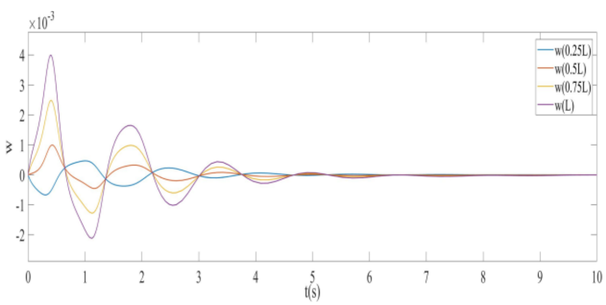

Figure 10. The deflection of the four equi-distanced points on the link without control

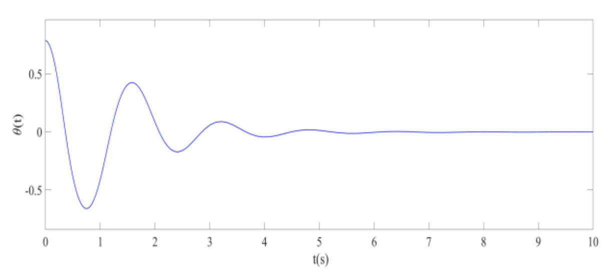

Figure 11. The angular position of flexible manipulator without control

6. Conclusion

All in all, for the proposed manipulator which was affected by time-varying boundary disturbance with rigid body nonlinear enormous angular position and translation, dynamic PDE-ODE delineation has been extracted grounded in Hamiltonian.

A novel boundary control method has been established grounded in the initial hybrid PDE-ODE dynamic modeling without any simplification to reach accurate locating of the system while all nonlinearities and gravitational force has been examined. Grounded in a newly introduced control strategy, by designing boundary control rules and boundary disturbance observer, transverse vibration and nonlinear angular position have been controlled with exponential decay rate, manipulator drove to the expected position, and at the same time the boundary disturbance has determined. The ultimate boundedness of the closed-loop system has been gained grounded in the Lyapunov direct method. Numerical simulations show the usefulness of the nonlinear proposed controller and observer.

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